# Linear and discret optimization

Made by http://hwdong.com

Notes from Friedrich Eisenbrand

## Feasible solutions

A point $x \in \mathbb{R}^n$ is called *feasible*, if $x$ satisfies all linear inequalities. If there are feasible solutions of a linear program, then the linear program is called *feasible*.

## Optimal solutions

A feasible $x \in \mathbb{R}^n$ is an *optimal solution* of ~~the~~ a linear program if $c^T x \geqslant c^T y$ for all feasible $y \in \mathbb{R}^n$.

## Bounded linear program

A linear program is *bounded* if there exists a constant $M \in \mathbb{R}$ such that $c^T x \leqslant M$ holds for all feasible $x \in \mathbb{R}^n$.

# Quiz

The linear program

$$\begin{array}{rrl} \max & x_1 & \\ \text{s.t.:} & x_1 + x_2 & \leqslant 1 \\ & x_1 & \geqslant 1 \end{array}$$

- is infeasible
- [✓ is feasible]
- is bounded unbounded.

$\forall k \geq 1 : (k, -k+1)$ is feas.

$M \in \mathbb{R}$

$k = \max\{M+1, 1\}$

$k \geq 1$ and

$(\underset{> M}{\overset{\uparrow}{k}}, -k+1)$ is feas.

# Quiz

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The linear program

$$\max\{c^T x : x \in \mathbb{R}^n, Ax = b\}$$

is feasible and unbounded if

- $b \in \ker(A)$ $\quad b \in \mathbb{R}^m \;, \ker(A) \subseteq \mathbb{R}^n$
- $b \in \mathrm{im}(A)$
- [✓ $b \in \mathrm{im}(A)$ and $c \in \ker(A) \setminus \underline{\{0\}}$]

$b \in \mathrm{im}(A) \Rightarrow \exists x^* \in \mathbb{R}^n$

s.t. $A \cdot x^* = b$

$$A \cdot (x^* + \underset{\substack{\uparrow \\ \in \mathbb{R}}}{\lambda} \cdot c) = \underbrace{A \cdot x^*}_{=b} + \underbrace{\lambda \cdot A \cdot c}_{=0} = b$$

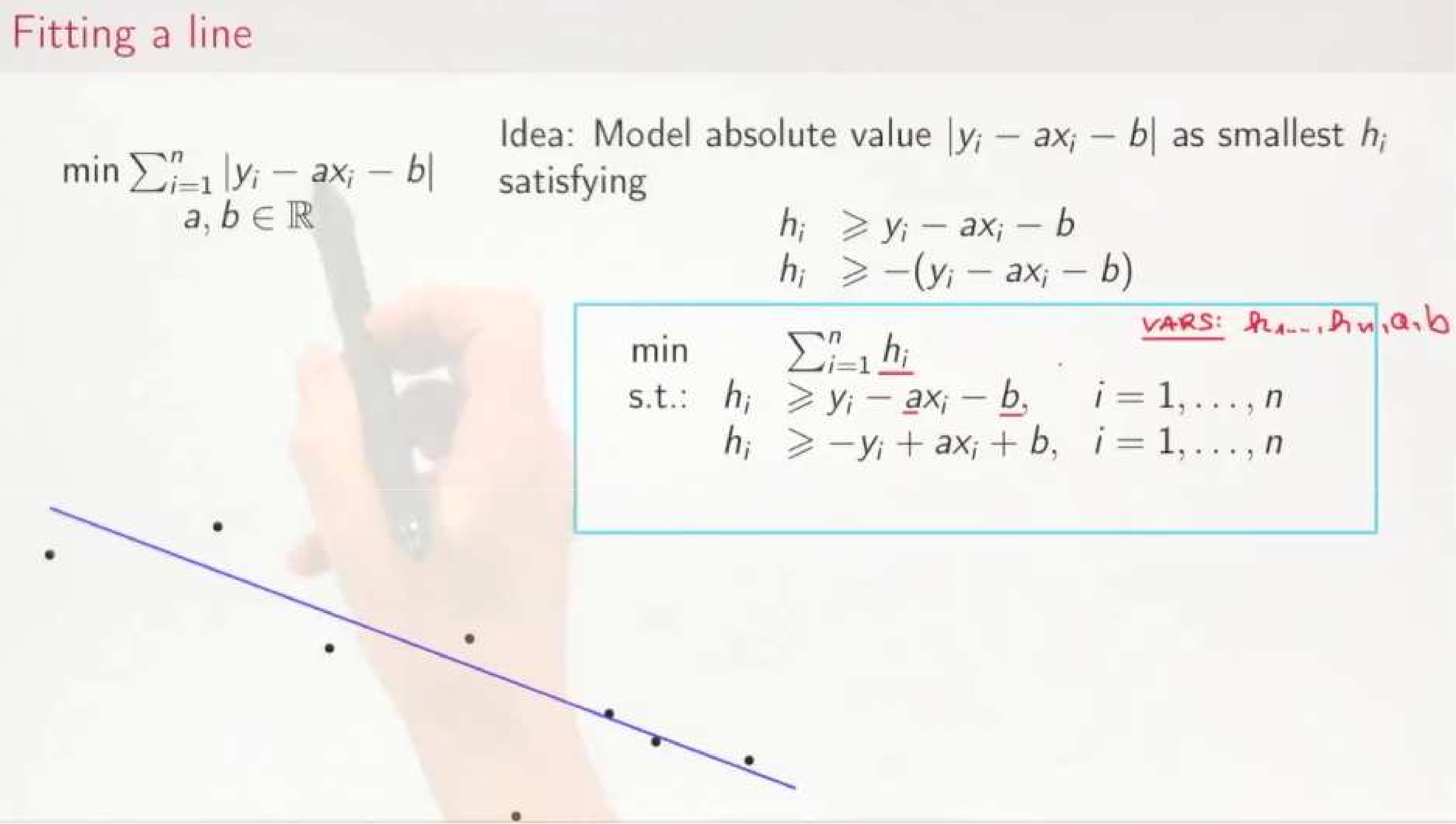
$\pi \in \mathbb{R}$:

$$c^T(x^* + \lambda \cdot c)$$

$$= c^T \cdot x^* + \lambda \cdot \underbrace{c^T \cdot c}_{>0}$$

$$\lambda \cdot c^T \cdot c + c^T \cdot x^* > \pi$$

$$\Leftrightarrow \quad \lambda > \frac{\pi - c^T \cdot x^*}{c^T \cdot c}$$



# Fitting a line

$$\min_{a,b\in\mathbb{R}} \sum_{i=1}^{n} |y_i - ax_i - b|$$

Idea: Model absolute value $|y_i - ax_i - b|$ as smallest $h_i$ satisfying

$$\begin{aligned} h_i &\geqslant y_i - ax_i - b \\ h_i &\geqslant -(y_i - ax_i - b) \end{aligned}$$

$\underline{\text{VARS:}}$ $h_1, \ldots, h_n, a, b$

$$\begin{array}{lll} \min & \sum_{i=1}^{n} \underline{h_i} & \\ \text{s.t.:} & h_i \geqslant y_i - \underline{a}x_i - \underline{b}, & i = 1, \ldots, n \\ & h_i \geqslant -y_i + ax_i + b, & i = 1, \ldots, n \end{array}$$

# Polyhedra

A set $P$ of vectors in $\mathbb{R}^n$ is a *polyhedron* if $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ for some matrix $A$ and some vector $b$.

$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Example:

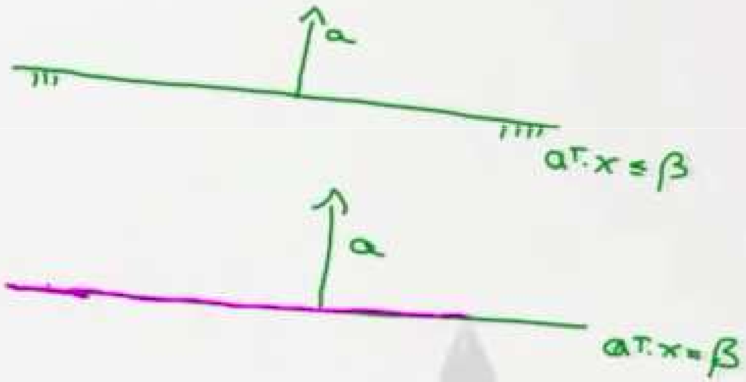
$P = \emptyset$

$\{x \in \mathbb{R}^n : \mathbf{0}^T x \leqslant -1\} = \emptyset$

$a \in \mathbb{R}^n \setminus \{0\}$ , $\beta \in \mathbb{R}$

$\{x \in \mathbb{R}^n : a^T \cdot x \leq \beta\}$ half space

$\{x \in \mathbb{R}^n : a^T \cdot x = \beta\}$ hyper plane



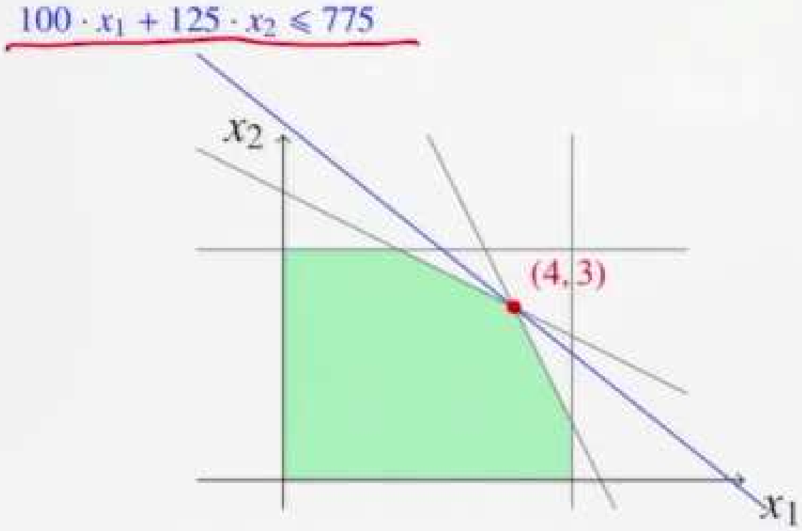
## Valid and active inequalities

An inequality $a^T x \leqslant \beta$ is *valid* for a polyhedron $P$ if each $x^* \in P$ satisfies $a^T x^* \leqslant \beta$.
An inequality $a^T x \leqslant \beta$ is *active* at $x^* \in \mathbb{R}^n$ if $a^T x^* = \beta$.

## Vertices

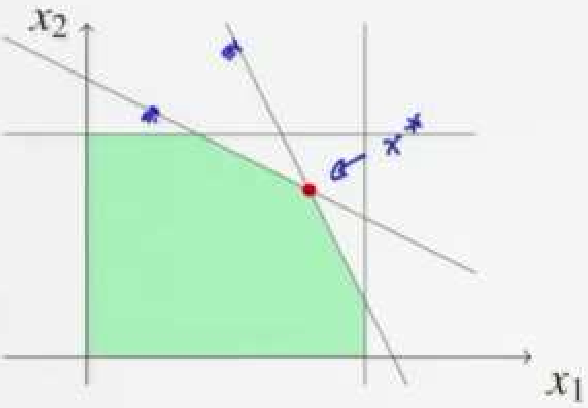
A point $x^* \in P$ is a *vertex* of $P$ if there exists an inequality $a^T x \leqslant \beta$ such that

- $a^T x \leqslant \beta$ is valid for $P$ and
- $a^T x \leqslant \beta$ is active at $x^*$ and not active at any other point in $P$.



$x^* \in P$ is a vertex $\Leftrightarrow$ $\exists\, c \in \mathbb{R}^n$ s.th.
$x^*$ is unique optimal solution of the
linear program $\max\{c^T \cdot x : x \in P\}$

# Alternative characterization of vertices: Intuition



$x^* \in P = \{x \in \mathbb{R}^n : Ax \leq b\}$

$\bar{A}x \leq \bar{b}$ active constraints

$x^*$ is the unique solution

$\bar{A}x = \bar{b}$

$\Updownarrow$

rank$(\bar{A}) = n$

$\Leftrightarrow$ columns of $\bar{A}$ are linearly independent

# Basic solutions

not necessarily feasible

Consider polyhedron $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$. A point $x^* \in \mathbb{R}^n$ is a *basic solution* if $\operatorname{rank}(A_I) = n$.

$A_I x \leq b_I$ sub-system of active ineq. at $x^*$

If $x^* \in P$, then $x^*$ is *basic feasible solution.*

Example: $P = \{x \in \mathbb{R}^3 : Ax \leqslant b\}$, $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \end{pmatrix}$, $x_1^* = \begin{pmatrix} -1/2 \\ 3/2 \\ 5/2 \end{pmatrix}$, $x_2^* = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

$A_1 X \leq b_1$

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

RANK $(A_1) = 3$

$x_1^*$ infeasible

basic sol.

$A_2 x \leq b_2$

$$A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \operatorname{rank}(A_2) = 3$$

$x_2^*$ feas. basic feasible solution

# Vertices and basic feasible solutions

if and only if

## Theorem

Let $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ and $x^* \in P$. Then $x^*$ is vertex of $P$ iff $x^*$ is basic feasible solution.

"$\Rightarrow$" $\underbrace{x^* \in P \text{ vertex}}$, assume not a basic feas. sol.

rank$(A_1) < n \Leftrightarrow$ Kernel$(A_1) \underset{\neq}{\supseteq} \{0\}$

$$Ax \leq b \begin{cases} \nearrow A_1 x \leq b_1 & (\text{active at } x^*) \quad A_1 x^* = b_1 \\ \searrow A_2 x \leq b_2 & (\text{inactive at } x^*) \quad A_2 x^* < b_2 \end{cases}$$

$\forall d \in \mathbb{R}^n \ \exists \varepsilon > 0$ s.th. $A_2(x^* \pm \varepsilon \cdot d) < b_2$

Vector
2 CHARACTERS
SMALL CAPS

$$A_1(x^* \pm \varepsilon \cdot d)$$
$$= \underbrace{A_1 x^*}_{= b_1} \pm \varepsilon \cdot \underbrace{A_1 \cdot d}_{= 0}$$

Let $d \in$ Ker$(A_1) \setminus \{0\}$

$$A_1(x^* \pm \varepsilon \cdot d) = \boxed{b_1}$$

$\exists c \in \mathbb{R}^n$ s.th. $x^*$ unique opt sol. of the LP $\max\{c^T \cdot x : x \in P\}$

$$c^T \cdot x^* > c^T\overbrace{(x^* + \varepsilon \cdot d)}^{\in P} \qquad \Rightarrow \qquad 0 > \varepsilon \cdot c^T \cdot d$$

$$c^T \cdot x^* > c^T\underbrace{(x^* - \varepsilon \cdot d)}_{\in P} \qquad\qquad 0 > -\varepsilon\, c^T \cdot d$$

↯

# Vertices and basic feasible solutions

**Theorem**

Let $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ and $x^* \in P$. Then $x^*$ is vertex of $P$ iff $x^*$ is basic feasible solution.

"$\Leftarrow$" $x^*$ basic feas. sol.

$$Ax \leq b \begin{cases} A_1 x \leq b_1 & \text{active at } x^* \\ A_2 x \leq b_2 & \text{inactive at } x^* \end{cases}$$

$\operatorname{rank}(A_1) = n$

$$\boxed{x^* \text{ unique sol of } A_1 x = b_1}$$

$\exists$ $\underline{c^T \cdot x \leq \delta}$ valid for $P$, active at $x^*$

an inactive at any $\underline{y^* \in P,\ y^* \neq x^*}$

$A_1 \cdot y^* \neq b_1$

$A_1 x \leq b_1 \cong \begin{matrix} a_1^T x \leq \beta_1 \\ \vdots \\ a_k^T x \leq \beta_k \end{matrix} \; + \quad A_1 \cdot y^* \leq b_1$

$c^T = (a_1^T + \dots + a_k^T)$, $\delta = (\beta_1 + \dots \dots + \beta_k)$

$c^T . x \leq \delta$ $\underline{\text{valid for } P}$ $\underline{\text{active at } x^*}$

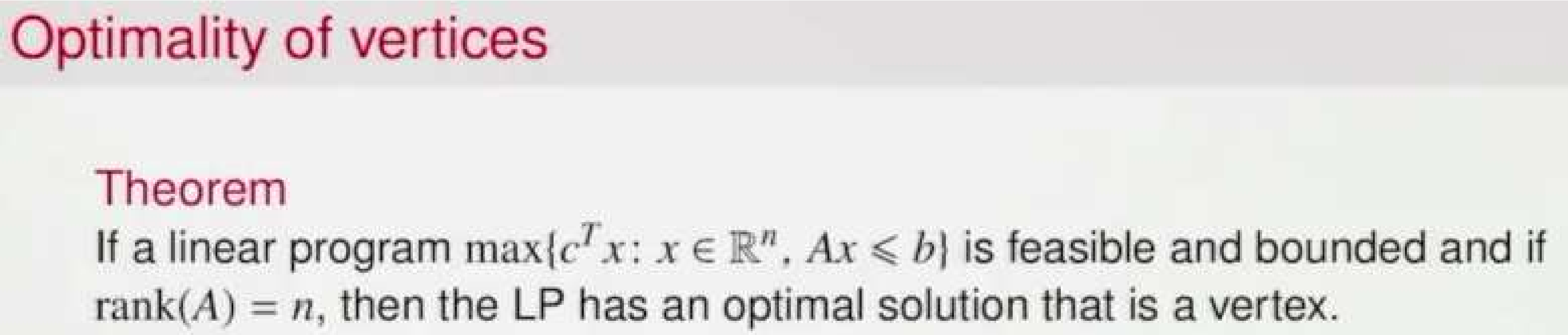
$\exists j$ s.th $a_j^T \cdot y^* < \beta_j$

$$c^T x^* = \sum_{i=1}^{k} a_i^T \cdot x^* = \sum_{i=1}^{k} \beta_i = \delta$$

$$c^T \cdot y^* = \sum_{i=1}^{k} a_i^T \cdot y^* = \sum_{\substack{i=1 \\ i \neq j}}^{k} \underbrace{a_i^T \cdot y^*}_{\leq \beta_i} + \underbrace{a_j^T \cdot y^*}_{< \beta_j} < \delta.$$

INACTIVE AT ANY OTHER $y^* \in P, y^* \neq x^*$

$\Rightarrow x^*$ vertex



## Optimality of vertices

**Theorem**

If a linear program $\max\{c^T x : x \in \mathbb{R}^n, Ax \leqslant b\}$ is feasible and bounded and if $\operatorname{rank}(A) = n$, then the LP has an optimal solution that is a vertex.

证明有点不太明白！

# Consequence: Restrict to vertices

$\text{MAX } c^T \cdot x$

$Ax \leq b$

$x \in \mathbb{R}^n$

Bounded

$\text{Rank}(A) = n$

$A \in \mathbb{R}^{m \times n}$

$\Rightarrow \exists$ vertex that is also opt. sol.

$x^*$ is vertex $\Rightarrow \exists\ B \subseteq \{1, \ldots, m\}$, s.th. , $|B| = n$

$x^*$ is unique solution of $A_B \cdot x = b_B$

- Enumerate all $B \subseteq \{1, \ldots, m\}$, $|B| = n$
- If $A_B$ is non-singular, $\underbrace{A_B^{-1} \cdot b_B}_{\text{feasible} \rightarrow \text{store it;}}$

Important consequence:

CAN BE SOLVED By enumerating all vertices and by picking the best one.

$$\left( \begin{array}{c} \boxed{m} \\ \boxed{n} \end{array} \right) = \frac{m!}{n!\,(m-n)!}$$

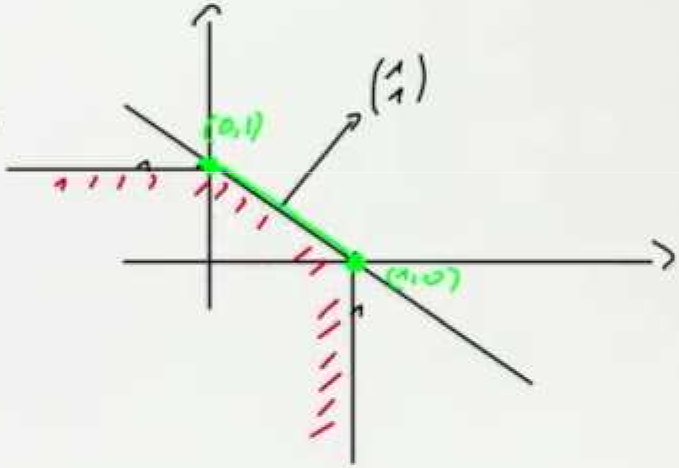
# Quiz

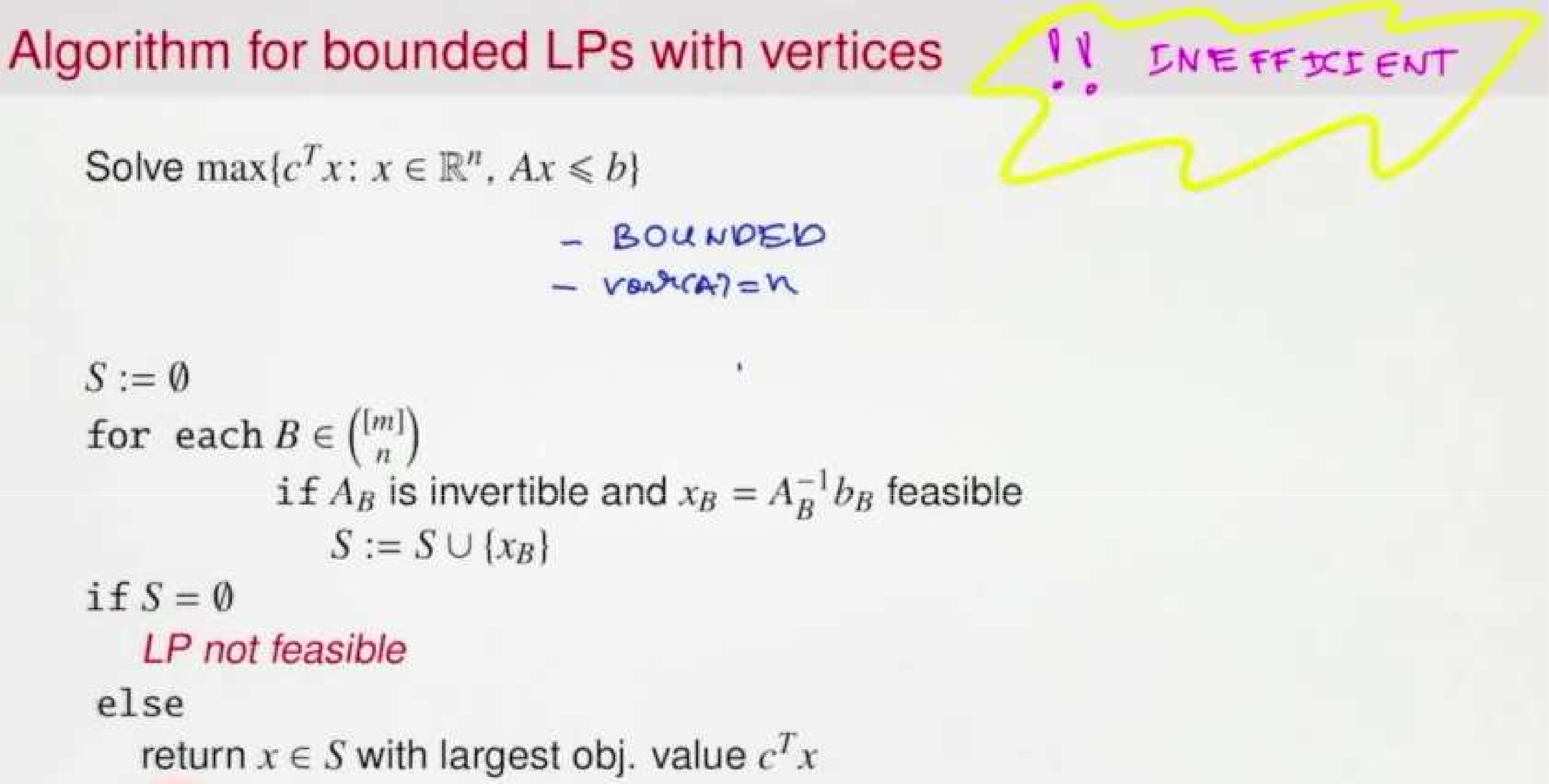
Consider

$$\begin{array}{rrcl} \max & x_1 + x_2 & & \\ & x_1 + x_2 & \leqslant & 1 \\ & x_1 & \leqslant & 1 \\ & x_2 & \leqslant & 1 \end{array}$$

Which of the following statements are true?

- Each optimal solution is a vertex.
- [✓ There exists an optimal solution that is a vertex.]
- [✓ There are infinitely many optimal solutions.]





# Algorithm for bounded LPs with vertices

!! INEFFICIENT

Solve $\max\{c^T x : x \in \mathbb{R}^n, Ax \leqslant b\}$

- BOUNDED
- rank(A) = n

```
$S := \emptyset$
for each $B \in \binom{[m]}{n}$
        if $A_B$ is invertible and $x_B = A_B^{-1} b_B$ feasible
            $S := S \cup \{x_B\}$
if $S = \emptyset$
    LP not feasible
else
    return $x \in S$ with largest obj. value $c^T x$
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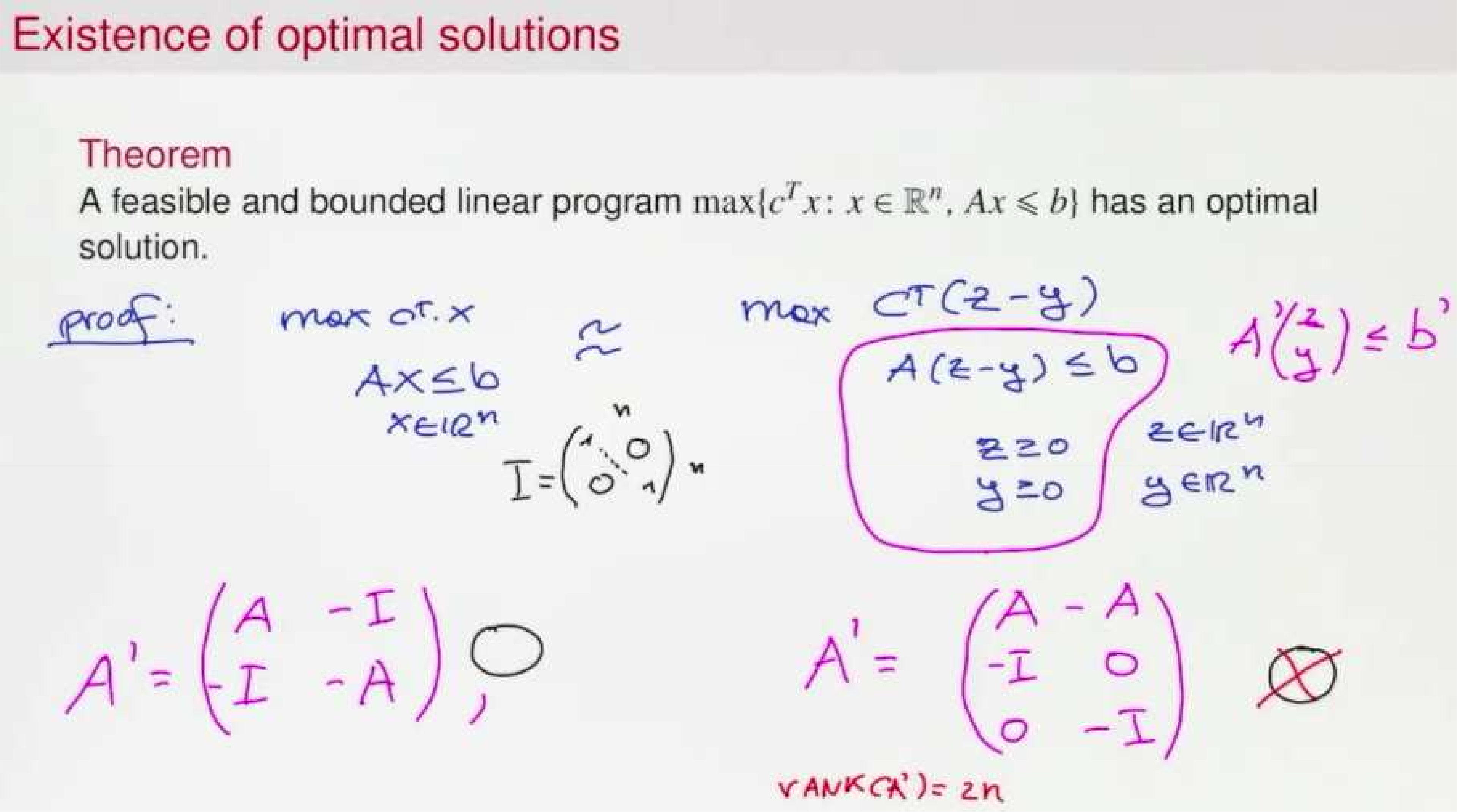


# Existence of optimal solutions

**Theorem**

A feasible and bounded linear program $\max\{c^T x : x \in \mathbb{R}^n, Ax \leqslant b\}$ has an optimal solution.

proof:

$$\max c^T \cdot x \quad \approx \quad \max c^T(z-y)$$

$$Ax \le b \qquad x \in \mathbb{R}^n$$

$$\boxed{\begin{array}{c} A(z-y) \le b \\ z \ge 0 \\ y \ge 0 \end{array}} \quad \begin{array}{l} z \in \mathbb{R}^n \\ y \in \mathbb{R}^n \end{array}$$

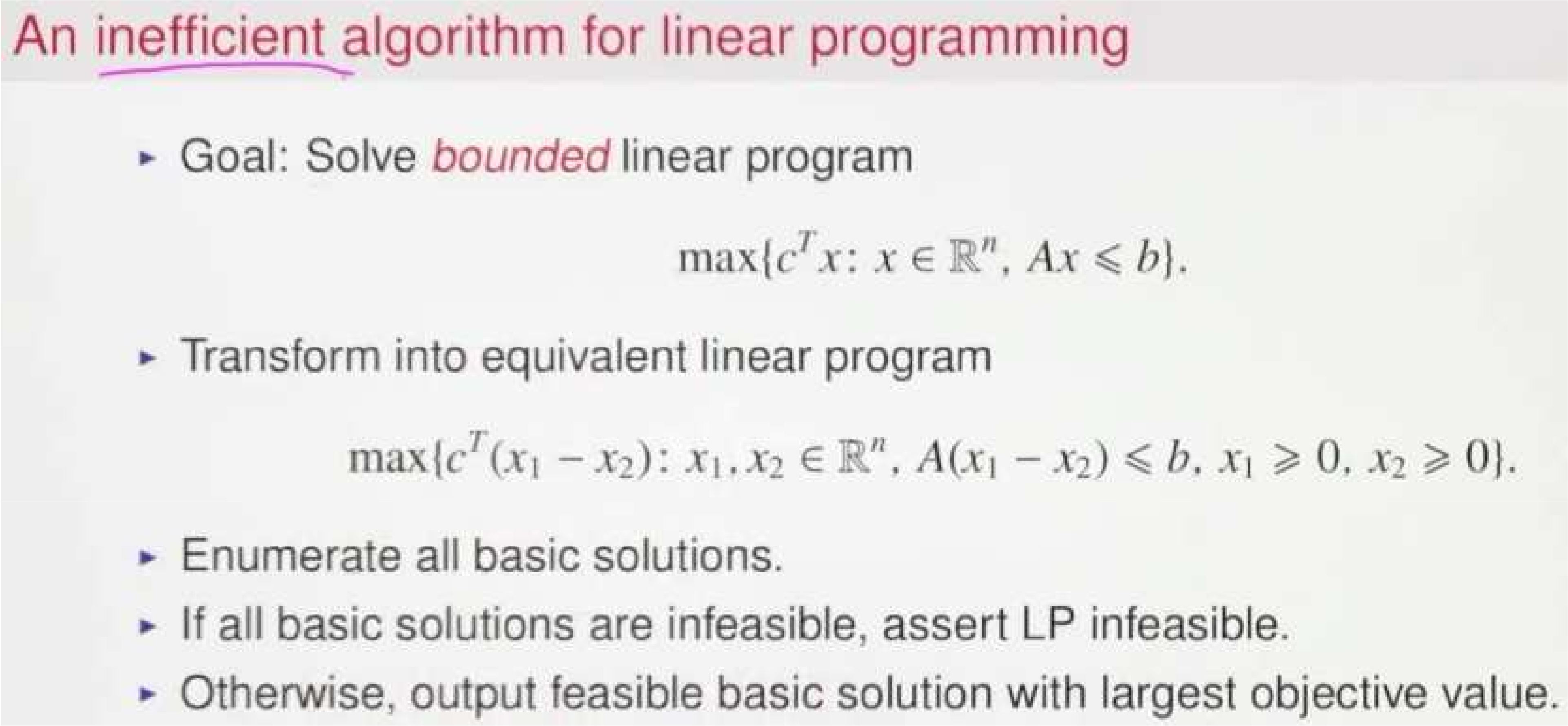
$$A'\begin{pmatrix} z \\ y \end{pmatrix} \le b'$$

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n \times n}$$

$$A' = \begin{pmatrix} A & -I \\ -I & -A \end{pmatrix}, \bigcirc$$

$$A' = \begin{pmatrix} A & -A \\ -I & 0 \\ 0 & -I \end{pmatrix} \quad \cancel{\otimes}$$

$\operatorname{rank}(A') = 2n$



# An inefficient algorithm for linear programming

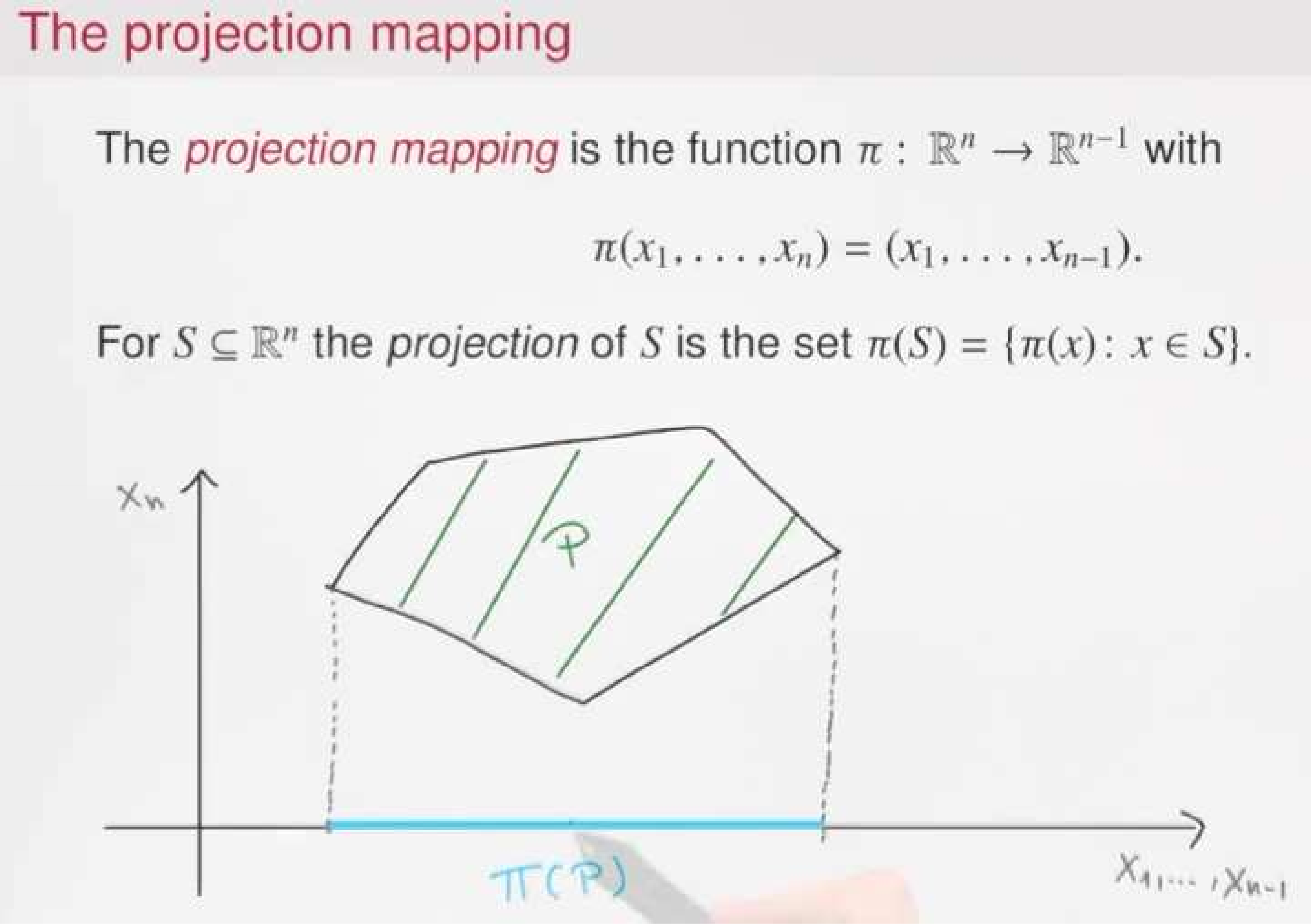
- Goal: Solve *bounded* linear program

$$\max\{c^T x : x \in \mathbb{R}^n, Ax \leqslant b\}.$$

- Transform into equivalent linear program

$$\max\{c^T(x_1 - x_2) : x_1, x_2 \in \mathbb{R}^n, A(x_1 - x_2) \leqslant b, x_1 \geqslant 0, x_2 \geqslant 0\}.$$

- Enumerate all basic solutions.
- If all basic solutions are infeasible, assert LP infeasible.
- Otherwise, output feasible basic solution with largest objective value.



# The projection mapping

The *projection mapping* is the function $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ with

$$\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}).$$

For $S \subseteq \mathbb{R}^n$ the *projection* of $S$ is the set $\pi(S) = \{\pi(x) \colon x \in S\}$.

# Completing a point in the projection

- Suppose we want to know whether $(x_1^*, \dots, x_{n-1}^*)$ is in $\pi(P)$ where $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$.
- Re-write each constraint $\sum_{j=1}^n a_{ij}x_j \leqslant b_i$ as

$I_> = \{i : a_{in} > 0\}$

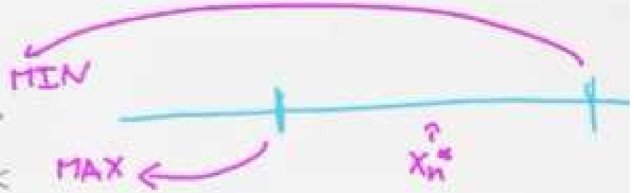
$J_< = \{j : a_{jn} < 0\}$

$$a_{in}x_n \leqslant -\sum_{j=1}^{n-1} a_{ij}x_j + b_i \qquad \left(* \frac{1}{a_{in}}\right)$$

- If $a_{in} \neq 0$ divide both sides by $a_{in}$. With $\overline{x} = (x_1, \dots, x_{n-1})$ we obtain an equivalent representation of $P$

$(x_1^*, \dots, x_{n-1}^*)$ can be completed

$$\begin{aligned} x_n^* &\leqslant d_i + f_i^T \overline{x}^* & i &\in I_> \\ x_n &\geqslant d_j + f_j^T \overline{x}^* & j &\in J_< \\ 0 &\leqslant d_k + f_k^T \overline{x}^* & k &\in K \end{aligned}$$



$$\Leftrightarrow \max_{j \in J_<} d_j + f_j^T \cdot \overline{x^*}$$

$$\leq \min_{i \in I_>} d_i + f_i^T \cdot \overline{x^*}$$

And $0 \leq d_k + f_k^T \overline{x^*} \quad \forall k \in K$

# The projection of a polyhedron

If $P \subseteq \mathbb{R}^n$ is represented by

$$\begin{array}{rcll} x_n & \leqslant & \underline{d_i + f_i^T \overline{x}} & i \in I_> \\ x_n & \geqslant & \underline{d_j + f_j^T \overline{x}} & j \in J_< \\ \rightarrow 0 & \leqslant & \underline{d_k + f_k^T \overline{x}} & k \in K \end{array}$$

then $\pi(P)$ is represented by

$$\left.\begin{array}{rcll} \underline{d_j + f_j^T \overline{x}} & \leqslant & \underline{d_i + f_i^T \overline{x}} & i \in I_>, j \in J_< \\ \rightarrow 0 & \leqslant & \underline{d_k + f_k^T \overline{x}} & k \in K \end{array}\right\} (1)$$

Describes Polyhedron $\subseteq \mathbb{R}^{n-1}$

VARS : $(x_1, \dots, x_{n-1}) = \overline{x}$

**Proof:** „$\subseteq$" $x^* = (x_1^*, \dots, x_n^*) \in P$

to show $\overline{x^*} = (x_1^*, \dots, x_{n-1}^*)$ sat. (1)

$$d_j + f_j^T \cdot \overline{x^*} \leq x_n^* = x_n^* \leq d_i + f_i^T \cdot \overline{x^*}$$

„$\supseteq$" $\overline{x^*} = (x_1^*, \dots, x_{n-1}^*)$ sot (1).

<u>To show:</u> $\exists \, \underline{x_n^*}$ s.th. $(x_1^*, \dots, x_{n-1}^*, x_n^*) \in P$

$$\boxed{\max_{j \in J_<} d_j + f_j^T \cdot \overline{x^*} \leq \min_{i \in I_>} d_i + f_i^T \cdot \overline{x^*}}$$

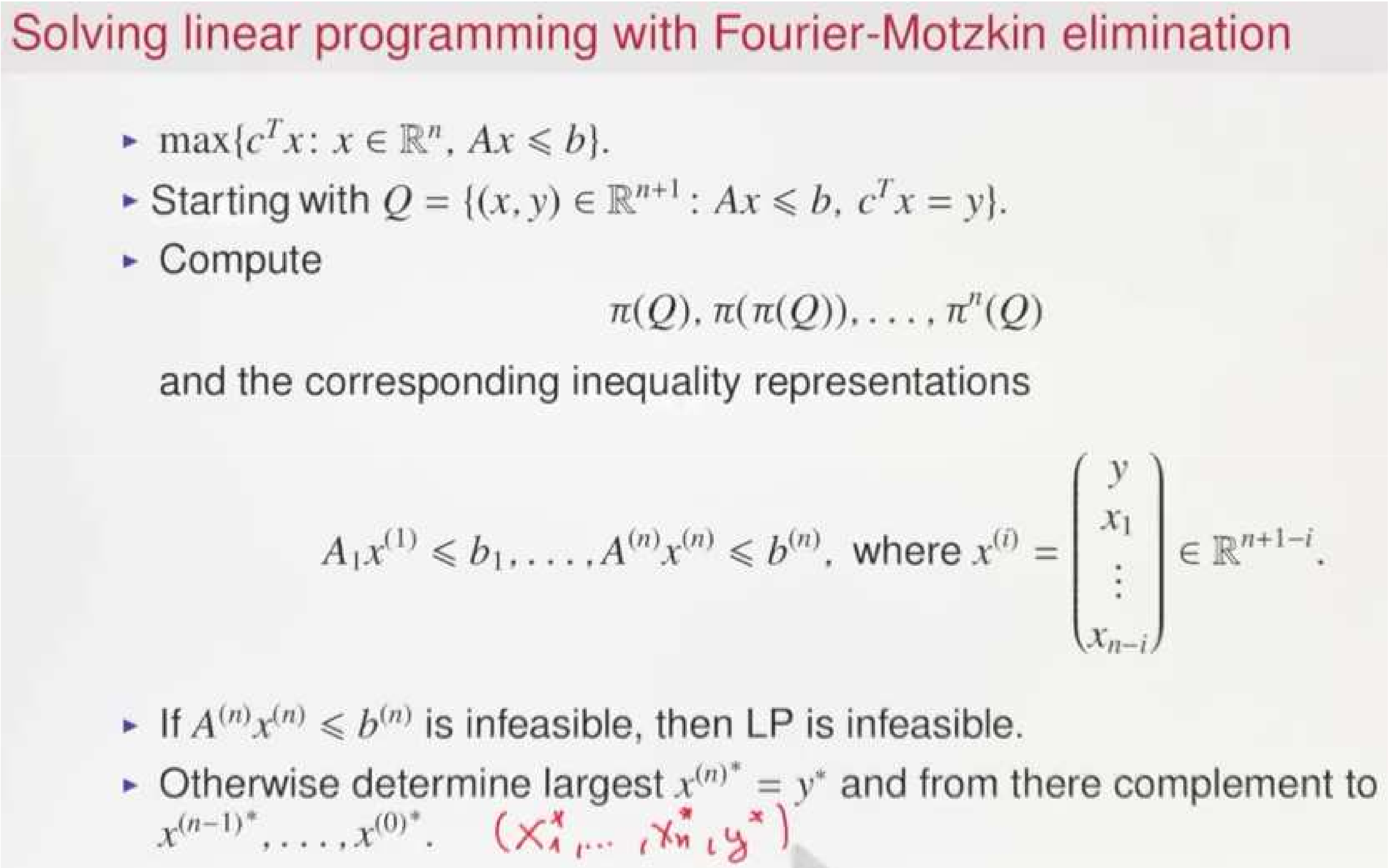
$\uparrow$

Clearly holds.

## The projection of a polyhedron (cont.)

### Corollary

If $P \subseteq \mathbb{R}^n$ is a polyhedron, then $\pi(P)$ is a polyhedron.



## Solving linear programming with Fourier-Motzkin elimination

- $\max\{c^T x : x \in \mathbb{R}^n, Ax \leqslant b\}$.
- Starting with $Q = \{(x, y) \in \mathbb{R}^{n+1} : Ax \leqslant b, c^T x = y\}$.
- Compute

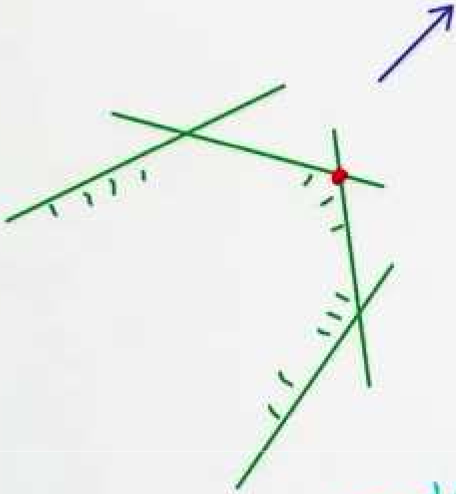
  $$\pi(Q), \pi(\pi(Q)), \dots, \pi^n(Q)$$

  and the corresponding inequality representations

  $$A_1 x^{(1)} \leqslant b_1, \dots, A^{(n)} x^{(n)} \leqslant b^{(n)}, \text{ where } x^{(i)} = \begin{pmatrix} y \\ x_1 \\ \vdots \\ x_{n-i} \end{pmatrix} \in \mathbb{R}^{n+1-i}.$$

- If $A^{(n)} x^{(n)} \leqslant b^{(n)}$ is infeasible, then LP is infeasible.
- Otherwise determine largest $x^{(n)^*} = y^*$ and from there complement to $x^{(n-1)^*}, \dots, x^{(0)^*}$. $(x_1^*, \dots, x_n^*, y^*)$

# Recap



$a_1^T \cdot x \le b_1$

$\vdots$

$a_m^T \cdot x \le b_m$

MAX $c^T \cdot x$

$Ax \le b$

$x \in \mathbb{R}^n$

$\text{rank}(A) = n$

bounded $\Rightarrow$ $\exists$ opt. sol that is also a vertex

$x^* \in P$ vertex $\Leftrightarrow$ subsystem of active constr. $\bar{A}x \le \bar{b}$ satisfies $\text{rank}(\bar{A}) = n$

$\Leftrightarrow$ $\exists B \subseteq \{1, \dots, m\}$, $|B| = n$ s.th. $A_B$ non-singular and $A_B \cdot x^* = b_B$ ($\Leftrightarrow x^* = A_B^{-1} b_B$)

$B = \{i_1, \dots, i_n\}$

$$A_B = \begin{pmatrix} a_{i_1}^T \\ \vdots \\ a_{i_n}^T \end{pmatrix}$$

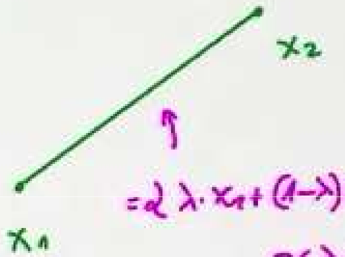
$$b_B = \begin{pmatrix} b_{i_1} \\ \vdots \\ b_{i_n} \end{pmatrix}$$

# Adjacent vertices

Two distinct vertices $x_1$ and $x_2$ of $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ are *adjacent*, if there exist $n-1$ *linearly independent inequalities* of $Ax \leqslant b$ active at both $x_1$ and $x_2$.

## Theorem

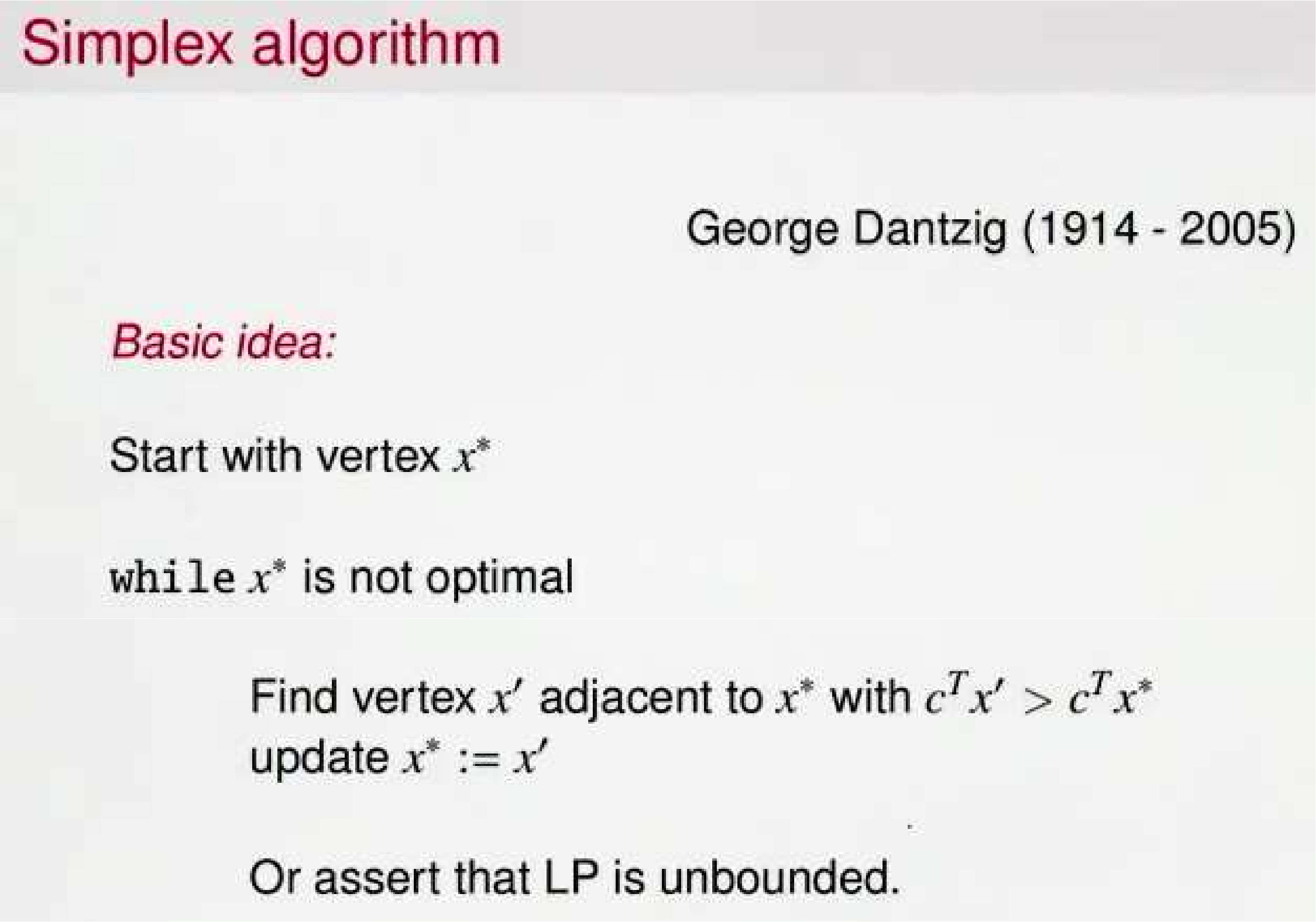
$x_1 \neq x_2 \in P$ are adjacent iff there exists $c \in \mathbb{R}^n$ such that set of optimal solutions of $\max\{c^T x : x \in P\}$ is $\underbrace{\{\lambda x_1 + (1-\lambda)x_2 : \lambda \in \mathbb{R}, 0 \leqslant \lambda \leqslant 1\}}_{\text{line segment spanned by } x_1 \text{ and } x_2}$.



$\uparrow$

$= \{\lambda \cdot x_1 + (1-\lambda) x_2 : \lambda \in \mathbb{R}, 0 \leq \lambda \leq 1\}$

Proof: Similar to proof of Vertex and Basic Feasible solution are equivalent concepts.



## Simplex algorithm

George Dantzig (1914 - 2005)

*Basic idea:*

```
Start with vertex $x^*$

while $x^*$ is not optimal

        Find vertex $x'$ adjacent to $x^*$ with $c^T x' > c^T x^*$
        update $x^* := x'$

        Or assert that LP is unbounded.
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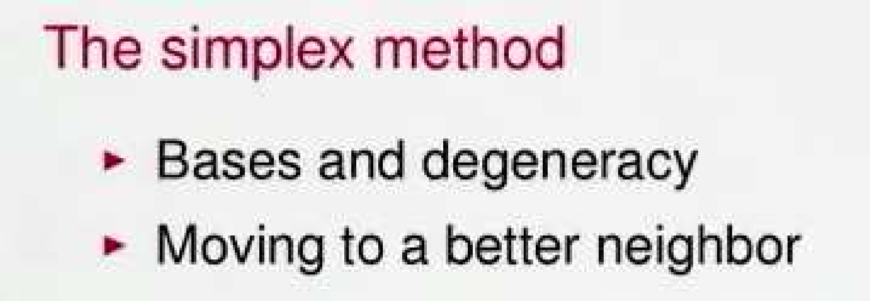


## The simplex method

- Bases and degeneracy
- Moving to a better neighbor

# Bases

A subset $B \subseteq \{1, \dots, m\}$ of the row-indices with $|B| = n$ and $A_B$ non-singular is called *basis* of the LP.

If in addition $A_B^{-1} b_B$ is feasible, then $B$ is called *feasible basis*.

$x^* \in P = \{x \in \mathbb{R}^n : Ax \le b\}$ is vertex $\iff$ $\exists\ \underline{B} \subseteq \{1, \dots, m\}$

s.th. $\underline{|B| = n,}$ $\underline{A_B \text{ non-singular}}$

and $x^* = A_B^{-1} \cdot b_B$

$$\boxed{\begin{array}{l} \text{MAX } c^T \cdot x \\ \quad x \in P \end{array}}$$

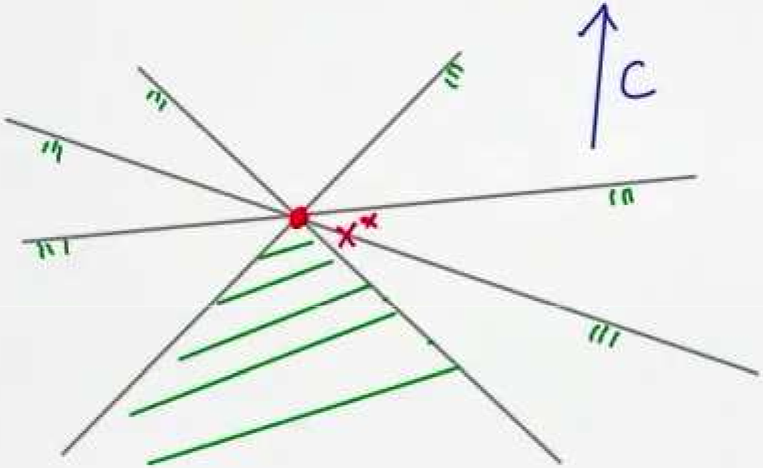
↖ LP

# Vertices and bases

A vertex $x^* \in P$ is represented by a basis $B$.

$x^* = A_B^{-1} \cdot b_B$

A vertex $x^*$ can be represented by several bases.



$\text{MAX } c^T \cdot x$

$Ax \leq b$

$x \in \mathbb{R}^2$

Quiz:

How many bases represent $x^*$?

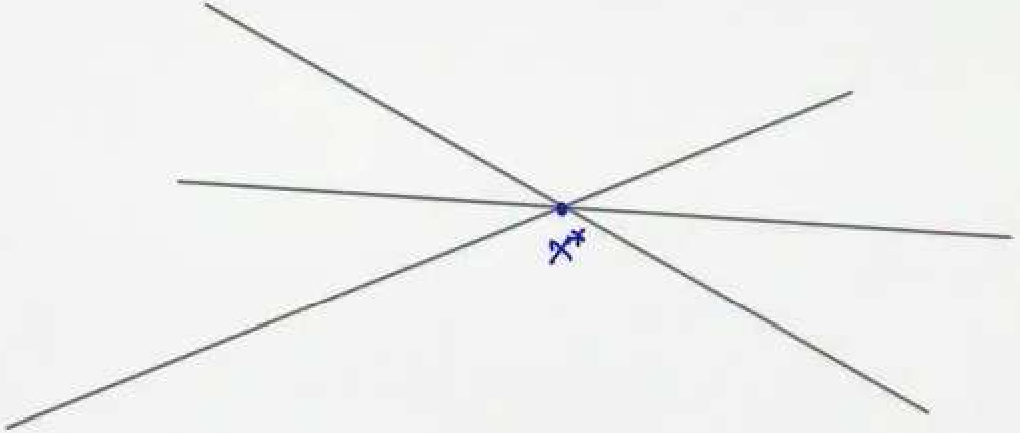
○ 2

○ 4

☒ 6

# Degeneracy

A linear program $\max\{c^T x \colon x \in \mathbb{R}^n,\, Ax \leqslant b\}$ is *degenerate* if there exists an $x^* \in \mathbb{R}^n$ such that there are more than $n$ constraints of $Ax \leqslant b$ that are active at $x^*$.



$n = 2$

3 Constraints active at $x^*$

# Optimal bases

A basis $B$ is called *optimal* if it is feasible and the unique $\lambda \in \mathbb{R}^m$ with

$$\lambda^T A = c^T \quad \text{and } \lambda_i = 0,\ i \notin B$$

satisfies $\lambda \geqslant 0$.

$\lambda_B^T \underline{A_B} = c^T$

$\lambda_B^T = c^T \cdot A_B^{-1}$

## Theorem

If $B$ is optimal basis, then $x^* = A_B^{-1} b_B$ is optimal solution of LP.

$a_1^T x \le b_1 \quad (*\lambda_1)$

$\vdots$

$a_m^T x \le b_m \quad (*\lambda_m)$

$+$

valid inequ.

$\Longrightarrow$

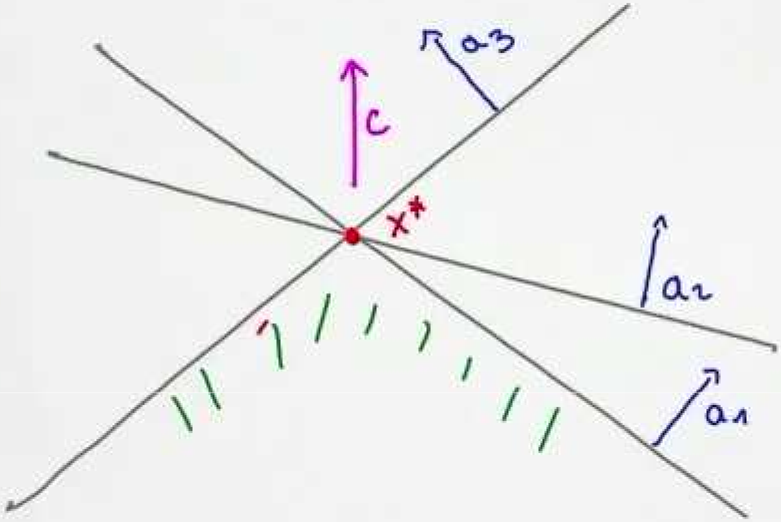
$$\underset{\boxed{c}^T}{\underset{\shortparallel}{\lambda^T \cdot A}}\, x \le \lambda^T \cdot b = \lambda_B^T \cdot \boxed{b_B} = A_B^{-1} \cdot x^*$$

$= \underline{A_B \cdot x^*}$

$$= \underset{\boxed{c}^T}{\underset{\shortparallel}{\underbrace{\lambda_B^T A_B}}} \cdot x^*$$

# Quiz

Which bases are optimal?



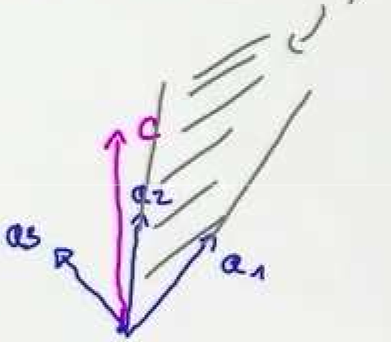
○ {1,2}

⊗ {1,3}

⊗ {2,3}

non-neg. linear comb. of a2 and a1

& c ∉



$\lambda^T_{\{1,2\}} \cdot A_{\{1,2\}} = c^T$

$\lambda_{\{1,2\}} \not\geq 0$

# The non-degenerate case

max $c^T.x$, $\quad Ax \le b$ $\begin{cases} A_B x \le b_B & \text{active at } x^* \\ A_{\bar B} x \le b_{\bar B} & \text{inactive.} \end{cases}$

**Theorem**

Suppose the LP is non-degenerate and $B$ is a feasible but not optimal basis, then $x^* = A_B^{-1} b_B$ is not an optimal solution.

$\lambda^T A = c^T$, $\lambda_j = 0 \ \forall j \notin B$, $\quad \lambda_i < 0$ for some $i \in B$.

Compute $d \in \mathbb{R}^n$ s.th. $A_{B \setminus \{i\}} \cdot d = 0$, $a_i^T \cdot d = -1$ $\quad$ ($A_B$ non-sing)

$$c^T \cdot d = \lambda_B^T \cdot A_B \cdot d = \underbrace{\lambda_i}_{<0} \cdot \underbrace{a_i^T \cdot d}_{=-1} \begin{matrix} > \\ < \\ \le \end{matrix} 0$$

For $\varepsilon > 0$ $\quad A_B(x^* + \varepsilon \cdot d) \begin{matrix} \le \\ = \\ \ne \end{matrix} b_B$

$b_B + \varepsilon \cdot \begin{pmatrix} 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow i$

$\exists\, \varepsilon^* > 0$ s.th.

$x^* + \varepsilon^* \cdot d$ is feasible.

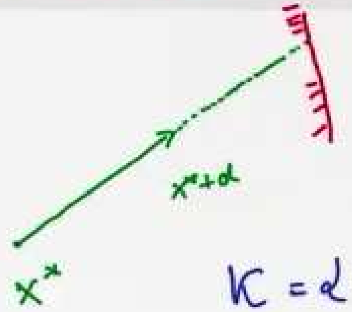
$$c^T(x^* + \varepsilon^* \cdot d) = c^T \cdot x^* + \underbrace{\varepsilon^* \cdot \underbrace{c^T \cdot d}_{>0}}_{>0}$$

# Moving to a better neighbor

LP NON-DEG.

- $B$ not an optimal basis
- $x^* = A_B^{-1} b_B$ corresponding basic feasible solution
- $\lambda_i < 0$ for some $i \in B$
- $a_j^T d = 0\,, j \in B \setminus \{i\}$
  $a_i^T d = -1$
- $c^T d > 0$
- there exists $\varepsilon > 0$ such that $x^* + \varepsilon d$ feasible

*Question:* How large can $\varepsilon$ be?



$a_1^T x \le b_1$

$\vdots$

$a_m^T \cdot x \le b_m$

$K = \{k : 1 \le k \le m,\ a_k^T \cdot d > 0\}$

CASE 1: $K = \emptyset$

LP UNBOUND

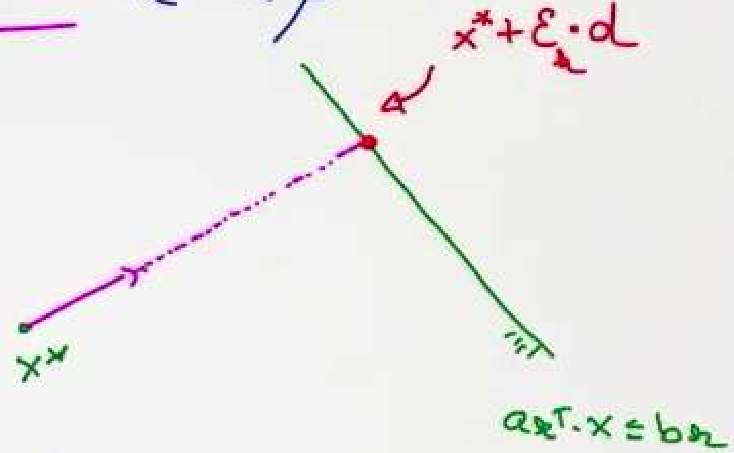
## Moving to a better neighbor

- $B$ not an optimal basis
- $x^* = A_B^{-1} b_B$ corresponding basic feasible solution
- $\lambda_i < 0$ for some $i \in B$
- $a_j^T d = 0\,, j \in B \setminus \{i\}$
  $a_i^T d = -1$
- $c^T d > 0$
- there exists $\varepsilon > 0$ such that $x^* + \varepsilon d$ feasible

*Question:* How large can $\varepsilon$ be?

CASE 2 : $K \neq \emptyset$

$k \in K$



$$a_k^T (x^* + \varepsilon_k \cdot d) = b_k$$

$$\Leftrightarrow \quad \varepsilon_k = \frac{b_k - a_k^T \cdot x^*}{\underbrace{a_k^T \cdot d}_{>0}}$$

$$\varepsilon^* = \min_{k \in K} \varepsilon_k$$

$k^* \in K$ Index where min is attained.

$x' = x^* + \varepsilon^* \cdot d$ , $B' = B \setminus \{i\} \cup \{k^*\}$ is BASIS

# Moving to a better neighbor

- $B$ not an optimal basis
- $x^* = A_B^{-1} b_B$ corresponding basic feasible solution
- $\lambda_i < 0$ for some $i \in B$
- $a_j^T d = 0\,, j \in B \setminus \{i\}$
  $a_i^T d = -1$
- $c^T d > 0$
- there exists $\varepsilon > 0$ such that $x^* + \varepsilon d$ feasible

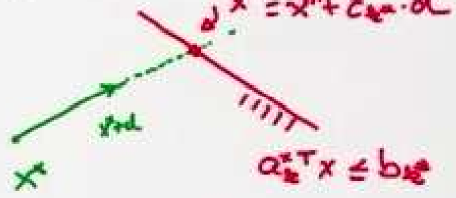
*Question:* How large can $\varepsilon$ be?

$B' = B \setminus \{i\} \cup \{k^*\}$ BASIS.

$d \perp a_j \;, j \in B \setminus \{i\}$

$d \not\perp a_{k^*} \;, d^T \cdot a_{k^*} > 0$

$\Rightarrow a_{k^*}$ IS NOT A LINEAR COMB. OF THE $a_j, \; j \in B \setminus \{i\}$.



The ineq. $a_j^T \cdot x \leq b_j \;, j \in B'$ are active at $x'$.

$\Rightarrow x'$ IS A VERTEX, ADJACENT TO $X^*$

# Simplex algorithm

George Dantzig (1914 - 2005)

*Basic idea:*

$\max \ c^T \cdot x$
$Ax \leq b$
$x \in \mathbb{R}^n$

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$

```
Start with vertex $x^*$

while $x^*$ is not optimal

    Find vertex $x'$ adjacent to $x^*$ with $c^T x' > c^T x^*$    $K \neq \emptyset$
    update $x^* := x'$

    Or assert that LP is unbounded.    $K = \emptyset$
```

# Simplex algorithm in basis notation

$\lambda^T \cdot A = c^T$ and $\lambda_j = 0 \;\; \forall j \notin B$

```
Start with feasible basis $B$
while $B$ is not optimal
        Let $i \in B$ be index with $\lambda_i < 0$
        Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0, \, j \in B \setminus \{i\}$ and $a_i^T d = -1$
        Determine $K = \{k \colon 1 \leqslant k \leqslant m, \, a_k^T d > 0\}$
        if $K = \emptyset$
            assert LP unbounded
        else
            Let $k \in K$ index where $\min_{k \in K} (b_k - a_k^T x^*) / a_k^T d$ is attained
            update $B := B \setminus \{i\} \cup \{k\}$
```

# The non-degenerate case

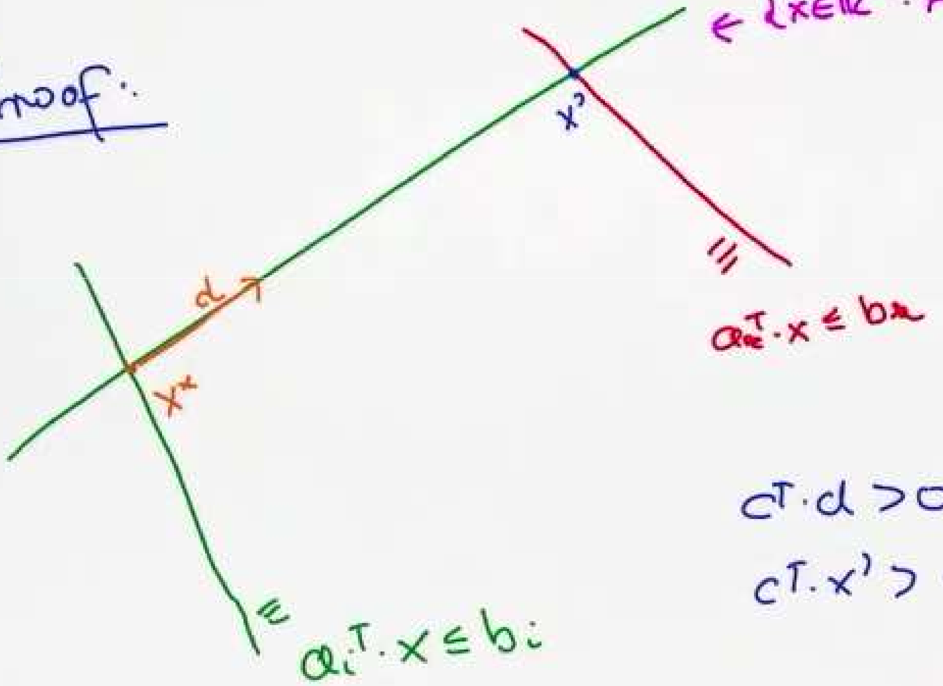
$$\begin{tikzcd} B \arrow[r] & B' \arrow[r] & B'' \arrow[r] & B''' \cdots\cdots & B^{(i)} & B^{(j)} \arrow[l, bend left] \end{tikzcd}$$

**Theorem**

If the linear program is non-degenerate, then the simplex algorithm terminates.

$\leftarrow \{x \in \mathbb{R}^n : A_{B\setminus\{i\}} x = b_{B\setminus\{i\}}\}$

Proof:



$c^T \cdot d > 0$

$c^T \cdot x' > c^T \cdot x^*$

$x^* = A_B^{-1} \cdot b_B$

$B' = B \setminus \{i\} \cup \{k\}$

$x' = A_{B'}^{-1} \cdot b_{B'}$

Each iteration:

Progress!

# Simplex algorithm: Bland's rule (Bland 1977)

LP DEGENERATE

```
Start with feasible basis $B$
while $B$ is not optimal
        Let $i \in B$ be smallest index with $\lambda_i < 0$
        Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0, j \in B \setminus \{i\}$ and $a_i^T d = -1$
        Determine $K = \{k \colon 1 \leqslant k \leqslant m, a_k^T d > 0\}$
        if $K = \emptyset$
            assert LP unbounded
        else
            Let $k \in K$ be smallest index where $\min_{k \in K} (b_k - a_k^T x^*) / a_k^T d$ is attained
            update $B := B \setminus \{i\} \cup \{k\}$
```

# Bland's rule avoids cycles

$\lambda,\quad \lambda^T \cdot A = c^T$

$d \quad c^T d > 0$

**Theorem**

If Bland's rule is applied, the simplex algorithm terminates.

Proof: Assume basis is re-visited

$j \in \{1, \dots, m\}$ largest index removed/added

$$\overset{\overset{u\ v}{\uparrow\downarrow}}{B_0} \quad B_1 \quad \overset{\uparrow}{B_2} \quad \cdots\cdots \quad \overset{\uparrow\downarrow}{B_{k-1}} \quad B_k$$

“Bland's rule”

$$\underbrace{B_0 \qquad\qquad B_k}_{=}$$

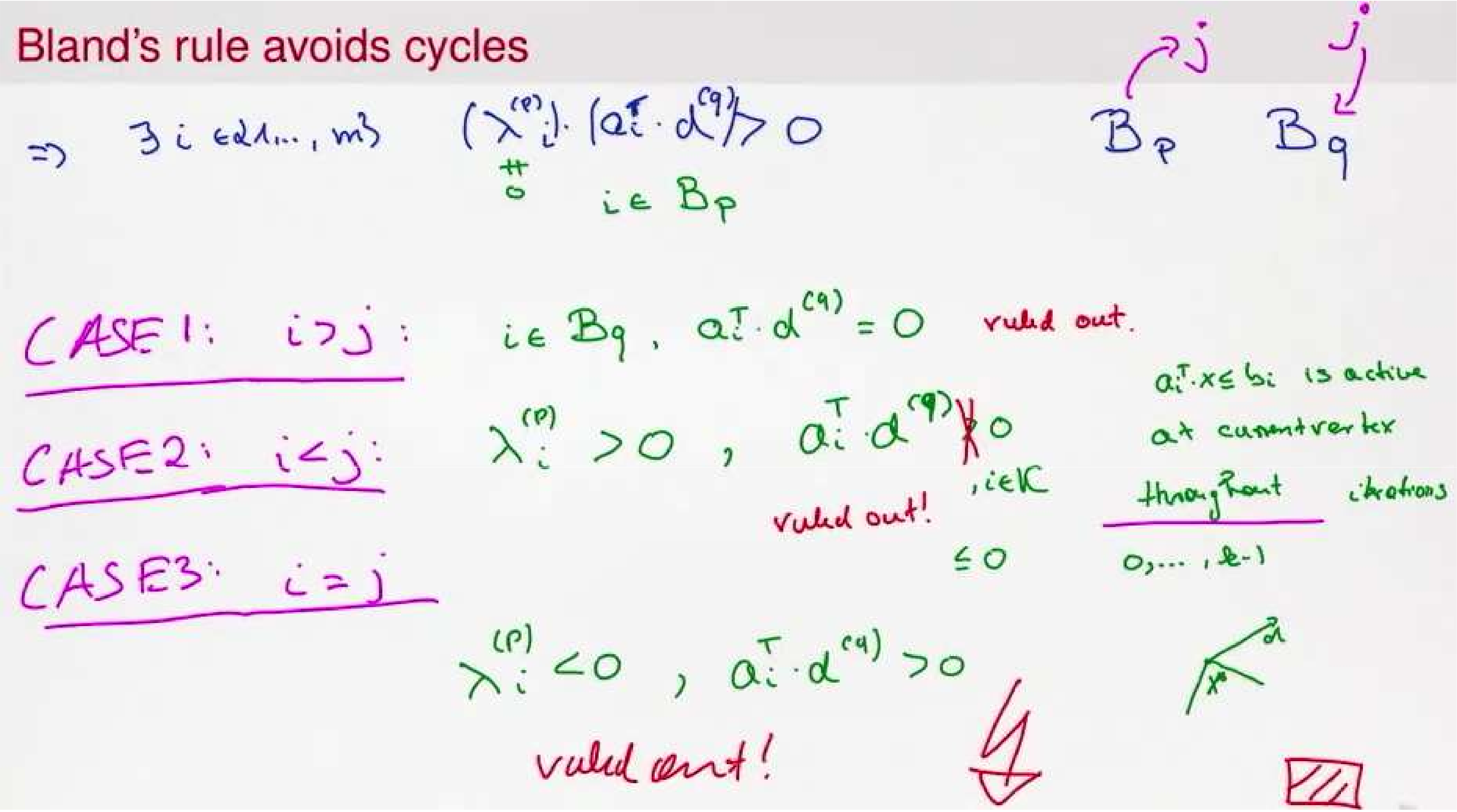
$$\overset{\to j}{B_p} \qquad \overset{\overset{j}{\downarrow}}{B_q}$$

$0 \le p < k$

$0 \le q < k$

$\lambda^{(p)T} \cdot A = c^T \quad , c^T \cdot d^{(q)} > 0$

$\Rightarrow \underline{(\lambda^{(p)})^T (A \cdot d)^{(q)} > 0}$



## Bland's rule avoids cycles

$\Rightarrow \quad \exists\, i \in \{1,\dots,m\} \quad (\lambda^{(p)}_i)\cdot(a_i^T \cdot d^{(q)}) > 0$

$\lambda^{(p)}_i \neq 0 \qquad i \in B_p$

$B_p \rightarrow j \qquad j \rightarrow B_q$

**CASE 1: $i > j$:** $\quad i \in B_q,\ a_i^T \cdot d^{(q)} = 0$ ruled out.

**CASE 2: $i < j$:** $\quad \lambda^{(p)}_i > 0,\ a_i^T \cdot d^{(q)} \cancel{>} 0$ $, i \in K$ $\le 0$ ruled out!

$a_i^T \cdot x \le b_i$ is active at current vertex throughout iterations $0,\dots,k-1$

**CASE 3: $i = j$**

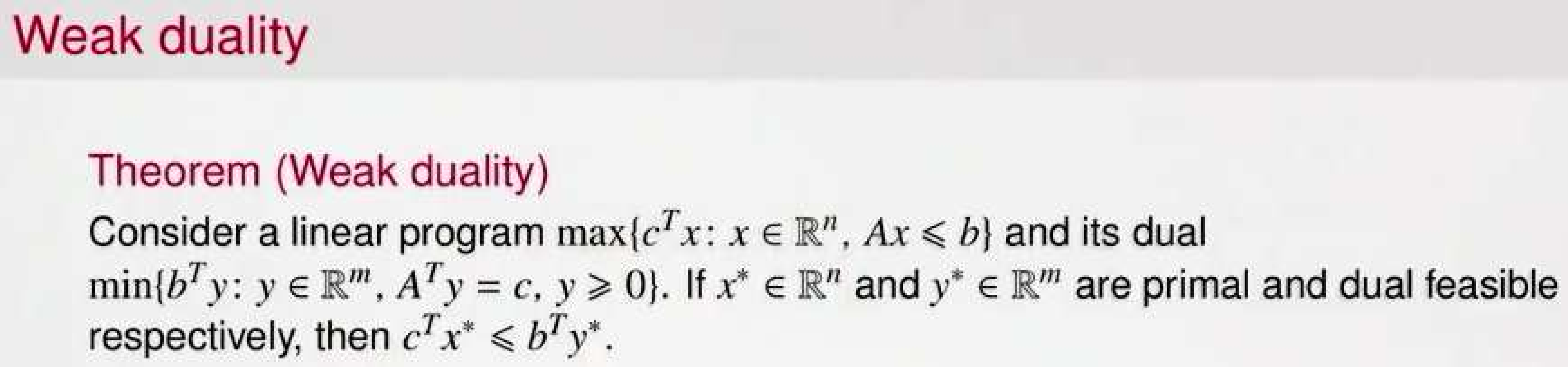
$\lambda^{(p)}_i < 0,\ a_i^T \cdot d^{(q)} > 0$

ruled out!

↯

▨

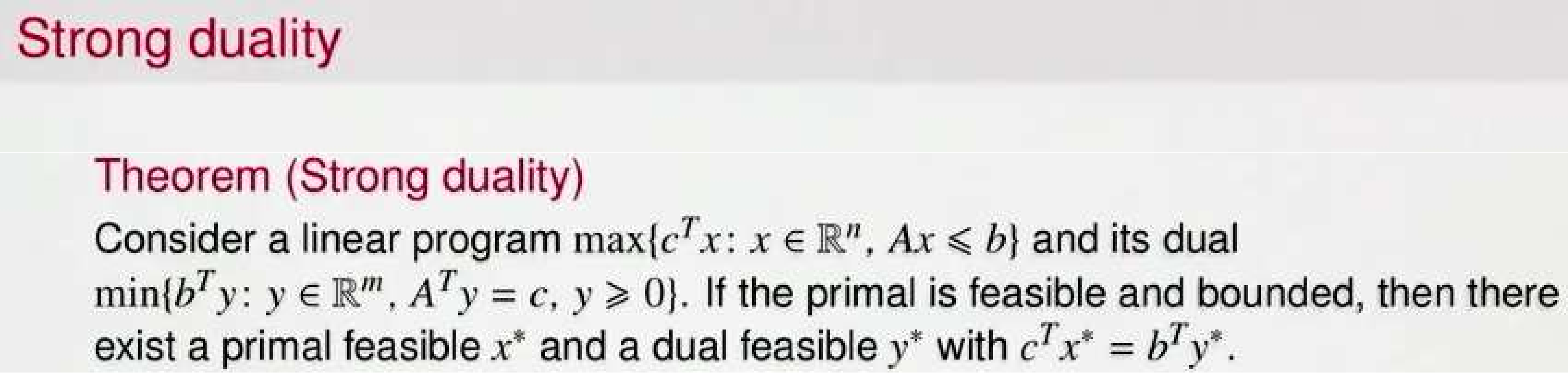
Case1 ,case2不太明白



## Weak duality

### Theorem (Weak duality)

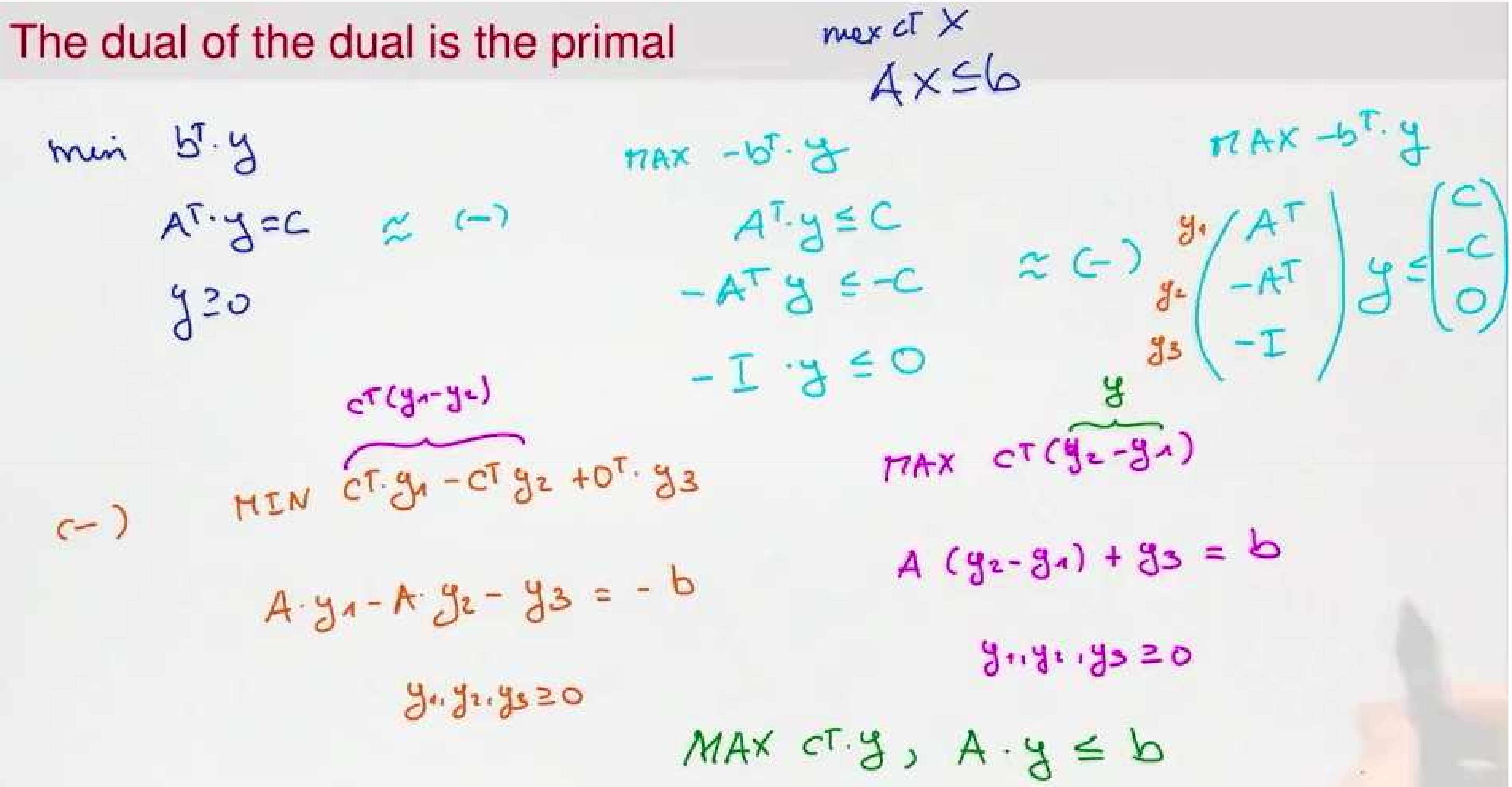
Consider a linear program $\max\{c^T x\colon x \in \mathbb{R}^n,\ Ax \leqslant b\}$ and its dual $\min\{b^T y\colon y \in \mathbb{R}^m,\ A^T y = c,\ y \geqslant 0\}$. If $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^m$ are primal and dual feasible respectively, then $c^T x^* \leqslant b^T y^*$.



## Strong duality

### Theorem (Strong duality)

Consider a linear program $\max\{c^T x\colon x \in \mathbb{R}^n,\ Ax \leqslant b\}$ and its dual $\min\{b^T y\colon y \in \mathbb{R}^m,\ A^T y = c,\ y \geqslant 0\}$. If the primal is feasible and bounded, then there exist a primal feasible $x^*$ and a dual feasible $y^*$ with $c^T x^* = b^T y^*$.



# The dual of the dual is the primal

max $c^T x$

$Ax \le b$

min $b^T \cdot y$

$A^T \cdot y = c$ ≈ (−)

$y \ge 0$

MAX $-b^T \cdot y$

$A^T \cdot y \le c$

$-A^T y \le -c$

$-I \cdot y \le 0$

≈ (−)

MAX $-b^T \cdot y$

$$\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\begin{pmatrix} A^T \\ -A^T \\ -I \end{pmatrix} y \le \begin{pmatrix} c \\ -c \\ 0 \end{pmatrix}$$

(−) MIN $\overbrace{c^T \cdot y_1 - c^T y_2}^{c^T(y_1 - y_2)} + 0^T \cdot y_3$

$A \cdot y_1 - A \cdot y_2 - y_3 = -b$

$y_1, y_2, y_3 \ge 0$

MAX $c^T(\overbrace{y_2 - y_1}^{y})$

$A\,(y_2 - y_1) + y_3 = b$

$y_1, y_2, y_3 \ge 0$

MAX $c^T \cdot y$, $A \cdot y \le b$

Which combinations are possible?

| P\ D | finite Opt | Unbounded | Infeasible |
|---|---|---|---|
| Finite Opt | ⊗ | ○ | ○ |
| Unbounded | ○ | ○ | ⊗ |
| Infeasible | ○ | ⊗ | possible |

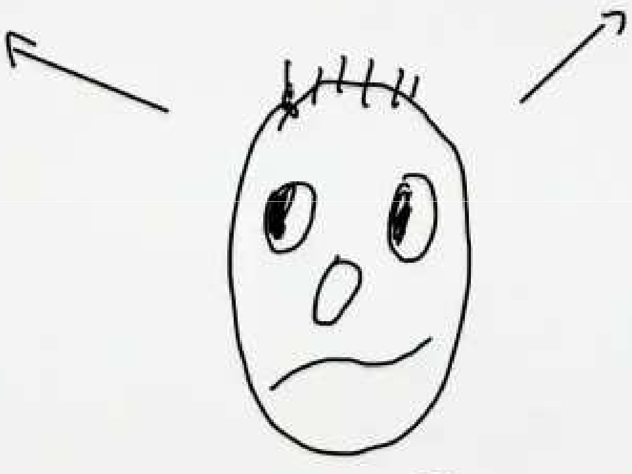
finite optimizaton: feasible and bound
unbound: 有feasible,但unbound
Infeasible:

# Proving optimality

**LP-solver 1**

feasible $x^* \in \mathbb{R}^n$

Says it's optimal



max $c^T \cdot x$

$Ax \leq b$

**LP-solver 2**

feasible $\underset{\overset{\uparrow}{(P)}}{x^*} \cdot \underset{\overset{\uparrow}{(D)}}{y^*}$

$c^T \cdot x^* = b^T \cdot y^*$

Proof of optimality

Simplex returns $x^*, y^*$

Size of $x^*$ and $y^*$ is polynomial in size of LP.

# Proving infeasibility

## Farkas' Lemma

A system of inequalities $Ax \leqslant b$ is infeasible if and only if there exists $\lambda \geqslant 0$ such that $\lambda^T A = 0$ and $\lambda^T b = -1$.

Proof: "⇐" $\underbrace{(\lambda^T \cdot A)}_{=0^T} x \leq \underbrace{\lambda^T \cdot b}_{=-1}$ valid ineq.

$\Rightarrow$ $Ax \leq b$ infeasible

"⇒" MAX $0^T \cdot x$ DUAL: MIN $b^T \cdot y$

$Ax \leq b$ $A^T \cdot y = 0$

$y \geq 0$

$y^* = 0$ feas. sol. $\Rightarrow$ DUAL feas.

DUAL UNBOUNDED!

$\exists y^* \geq 0,\ A^T \cdot y^* = 0,\ b^T \cdot y^* < 0$, $y' = y^* / |b^T \cdot y^*|$

$y' \geq 0$, $A^T \cdot y' = 0$, $b^T \cdot y' = \underbrace{b^T \cdot y^*}_{<0} / |b^T \cdot y^*| = -1$ $\lambda = y'$ ▨

# Discret optimization

# Bipartite graphs

A graph $G = (V, E)$ is *bipartite*, if one can partition $V$ into $V = A\dot{\cup}B$ such that each edge $e \in E$ satisfies $|e \cap A| = |e \cap B| = 1$.

# Matchings

A *matching* is a subset $M \subseteq E$ of the edges such that each $e_1 \neq e_2 \in M$ satisfy $e_1 \cap e_2 = \emptyset$.

The edges in a matching “do not touch”.

# The maximum weight (bipartite) matching problem

Given a (bipartite) graph $G = (V, E)$ and *edge weights* $w : E \to \mathbb{N}_0$, determine a matching $M \subseteq E$ such that

$$w(M) := \sum_{e \in M} w_e \quad \text{is maximal.}$$

## $w$-vertex covers

Let $G = (V, E)$ be a graph with edge weights $w : E \to \mathbb{N}_0$. A *$w$-vertex cover* is a vector $y \in \mathbb{N}_0^{|V|}$ such that

$$\forall uv \in E : \quad y_u + y_v \geqslant w_{uv}.$$

The *value* of a $w$-vertex cover $y$ is $\sum_{v \in V} y_v$.

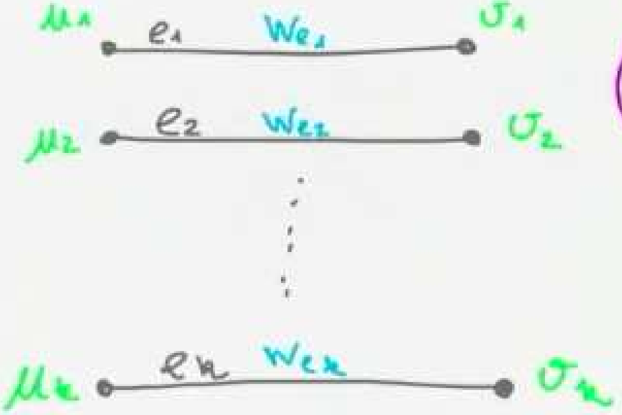
**Lemma (Weak duality)**

Let $G = (V, E)$ be a graph and let $w : E \to \mathbb{N}_0$ be edge-weights. If $M$ is a matching of $G$ and if $y$ is a $w$-vertex cover of $G$, then

$$w(M) \leqslant \sum_{v \in V} y_v.$$

Weight of M.

Proof:

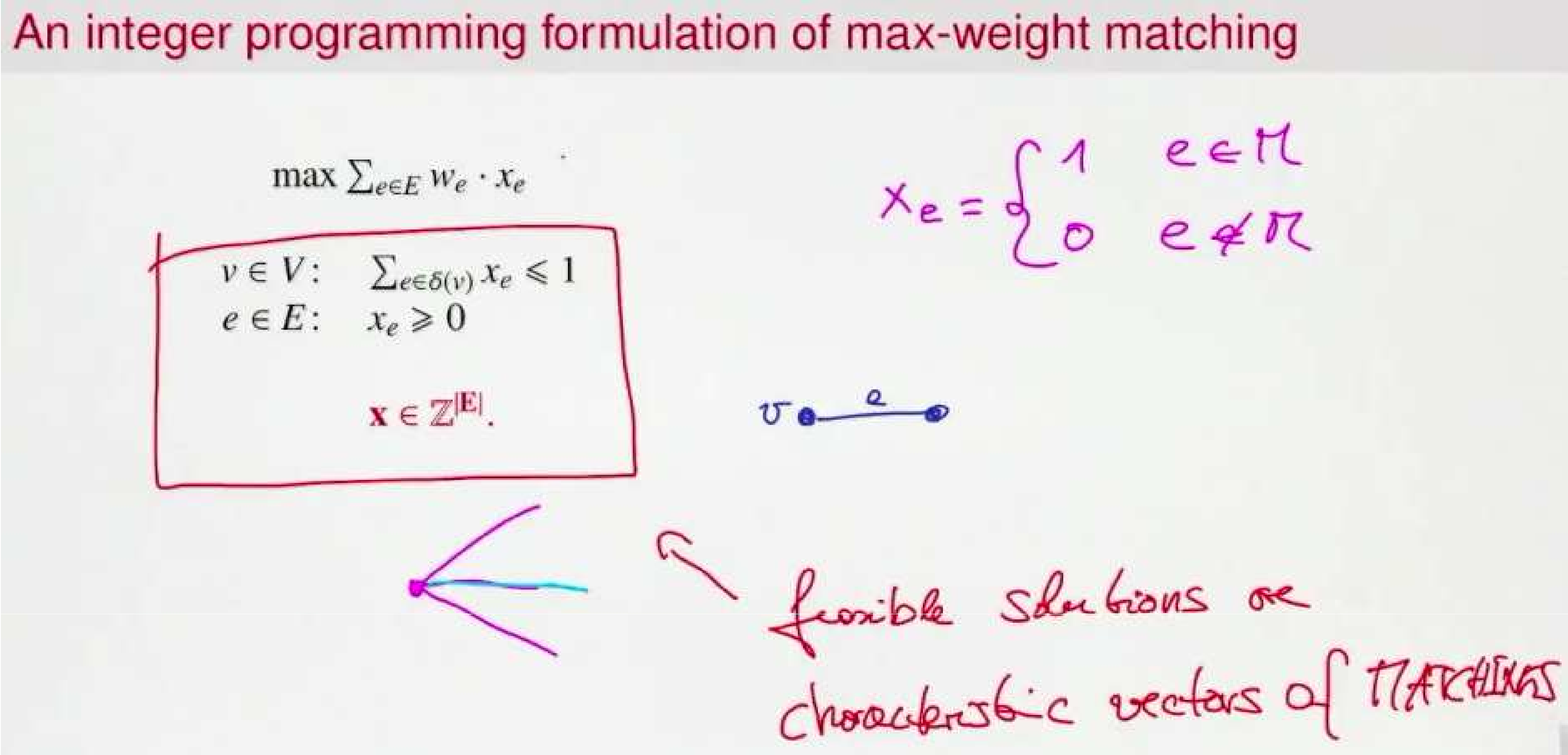


$$M = \{e_1, e_2, \cdots, e_k\}$$

$$\boxed{\sum_{i=1}^{k} \underbrace{w_{e_i}}_{\leq y_{u_i} + y_{v_i}}}$$

$$\leq \sum_{v \in V} y_v \leftarrow$$

VAL. OF W-VERTEX COVER



## An integer programming formulation of max-weight matching

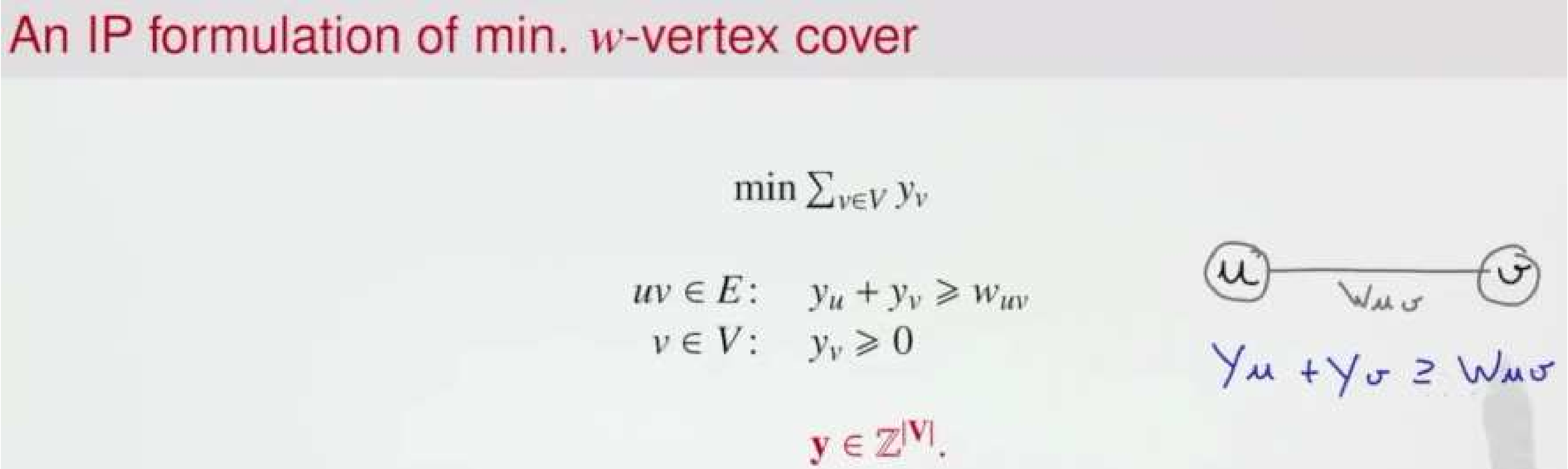
$$\max \sum_{e\in E} w_e \cdot x_e$$

$$v \in V: \quad \sum_{e\in\delta(v)} x_e \leqslant 1$$
$$e \in E: \quad x_e \geqslant 0$$

$$\mathbf{x} \in \mathbb{Z}^{|\mathbf{E}|}.$$

$$x_e = \begin{cases} 1 & e \in M \\ 0 & e \notin M \end{cases}$$

feasible solutions are characteristic vectors of MATCHINGS



## An IP formulation of min. $w$-vertex cover

$$\min \sum_{v\in V} y_v$$

$$uv \in E: \quad y_u + y_v \geqslant w_{uv}$$
$$v \in V: \quad y_v \geqslant 0$$

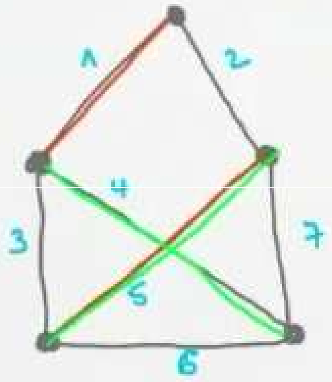
$$\mathbf{y} \in \mathbb{Z}^{|\mathbf{V}|}.$$

$$y_u + y_v \geq w_{uv}$$

# Towards a second proof of weak duality via LP-duality

## Idea

Describe *characteristic* vectors $\chi^M$ of matchings by linear constraints and the *integrality* constraint.



Enumerate edges

Matching can be described as 0/1-vector

$$\chi^M = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\chi^M = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Quiz :

IS $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ a characteristic vector of a matching

YES ○ NO ☒

# The description

For $v \in V$ we denote the set of edges *incident* to $v$ by

$$\delta(v) = \{e \in E \colon v \in e\}.$$

The set $\{\chi^M : M$ matching of $G\}$ is the set of *feasible solutions of*

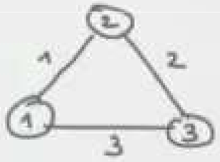
$$\begin{array}{ll} v \in V: & \sum_{e \in \delta(v)} x_e \leqslant 1 \\ e \in E: & x_e \in \{0,1\}. \end{array}$$

Quiz:

⊗

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$x \in \{0,1\}^3$



○

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$x \in \{0,1\}^3$

# Proving weak duality via LP duality

**Theorem**

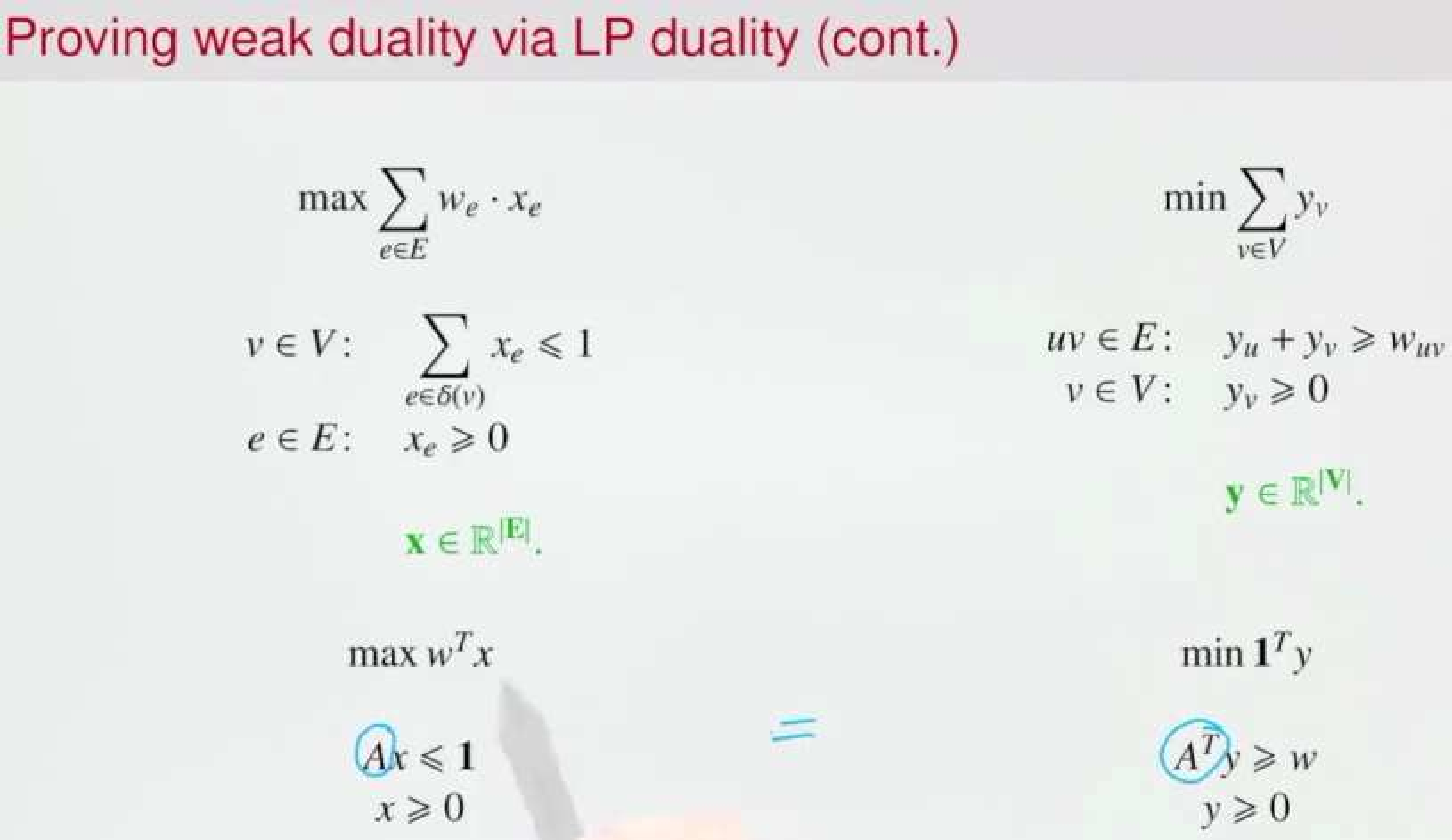
The max. weight of a matching is at most the min. value of a $w$-vertex cover.

$$\begin{array}{ccccccc}
\begin{array}{ll} & \max \sum_{e\in E} w_e \cdot x_e \\ v \in V: & \sum_{e\in\delta(v)} x_e \leqslant 1 \\ e \in E: & x_e \geqslant 0 \\ & \mathbf{x} \in \mathbb{Z}^{|\mathbf{E}|}. \end{array}
& \leq &
\begin{array}{ll} & \max \sum_{e\in E} w_e \cdot x_e \\ v \in V: & \sum_{e\in\delta(v)} x_e \leqslant 1 \\ e \in E: & x_e \geqslant 0 \\ & \mathbf{x} \in \mathbb{R}^{|\mathbf{E}|}. \end{array}
& = &
\begin{array}{ll} & \min \sum_{v\in V} y_v \\ uv \in E: & y_u + y_v \geqslant w_{uv} \\ v \in V: & y_v \geqslant 0 \\ & \mathbf{y} \in \mathbb{R}^{|\mathbf{V}|}. \end{array}
& \leq &
\begin{array}{ll} & \min \sum_{v\in V} y_v \\ uv \in E: & y_u + y_v \geqslant w_{uv} \\ v \in V: & y_v \geqslant 0 \\ & \mathbf{y} \in \mathbb{Z}^{|\mathbf{V}|}. \end{array}
\end{array}$$

↑ MAX MATCHING

↑ MIN. VAL. W-VERTEX COVER

$$\text{MAX MATCHING} \leq \text{MIN. VAL. W-VERTEX COVER}$$



# Proving weak duality via LP duality (cont.)

$$\max \sum_{e \in E} w_e \cdot x_e \qquad\qquad \min \sum_{v \in V} y_v$$

$$v \in V: \quad \sum_{e \in \delta(v)} x_e \leqslant 1 \qquad\qquad uv \in E: \quad y_u + y_v \geqslant w_{uv}$$

$$e \in E: \quad x_e \geqslant 0 \qquad\qquad v \in V: \quad y_v \geqslant 0$$

$$\mathbf{x} \in \mathbb{R}^{|\mathbf{E}|}. \qquad\qquad \mathbf{y} \in \mathbb{R}^{|\mathbf{V}|}.$$

$$\max w^T x \qquad\qquad \min \mathbf{1}^T y$$

$$Ax \leqslant \mathbf{1} \qquad = \qquad A^T y \geqslant w$$

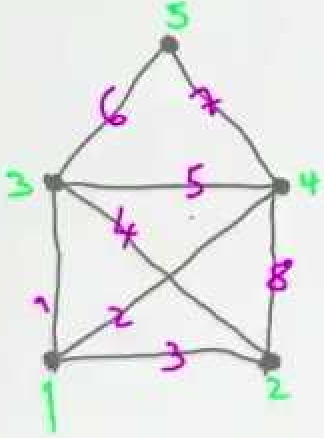
$$x \geqslant 0 \qquad\qquad y \geqslant 0$$

# The node-edge incidence matrix

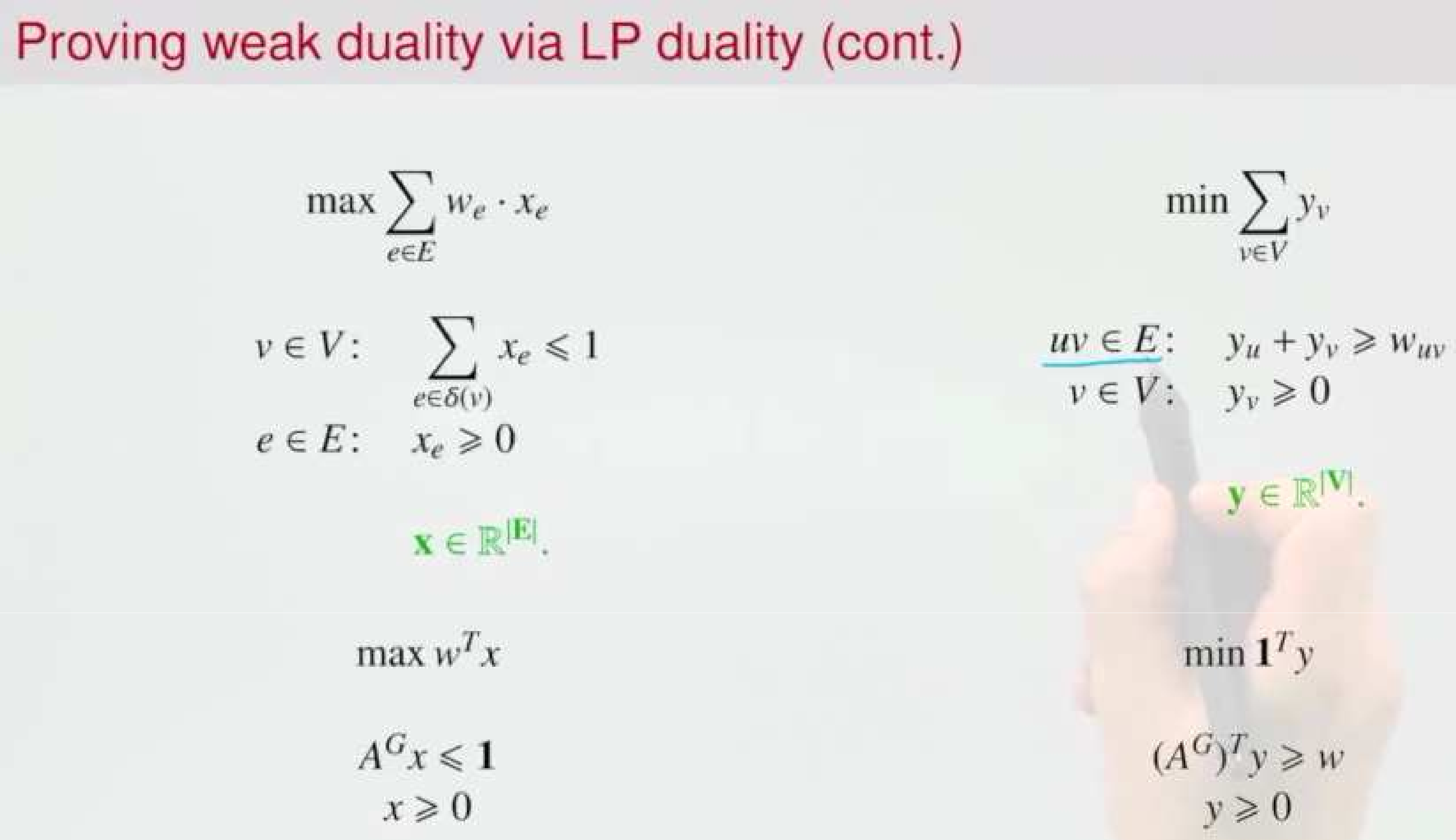
Let $G = (V, E)$ be a graph and suppose the nodes and edges are ordered as $v_1, \dots, v_n$ and $e_1, \dots, e_m$. The matrix $A^G \in \{0,1\}^{n \times m}$ with

$$A^G_{i,j} = \begin{cases} 1 & \text{if } v_i \in e_j, \\ 0 & \text{otherwise} \end{cases}$$

is the *node-edge incidence* matrix of $G$.



$$A_G = \begin{array}{c|c|c|c|c|c|c|c|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 3 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \hline 4 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ \hline 5 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$



## Proving weak duality via LP duality (cont.)

$$\max \sum_{e \in E} w_e \cdot x_e$$

$$\begin{aligned} v \in V: &\quad \sum_{e \in \delta(v)} x_e \leqslant 1 \\ e \in E: &\quad x_e \geqslant 0 \end{aligned}$$

$$\mathbf{x} \in \mathbb{R}^{|\mathbf{E}|}.$$

$$\min \sum_{v \in V} y_v$$

$$\begin{aligned} \underline{uv \in E}: &\quad y_u + y_v \geqslant w_{uv} \\ v \in V: &\quad y_v \geqslant 0 \end{aligned}$$

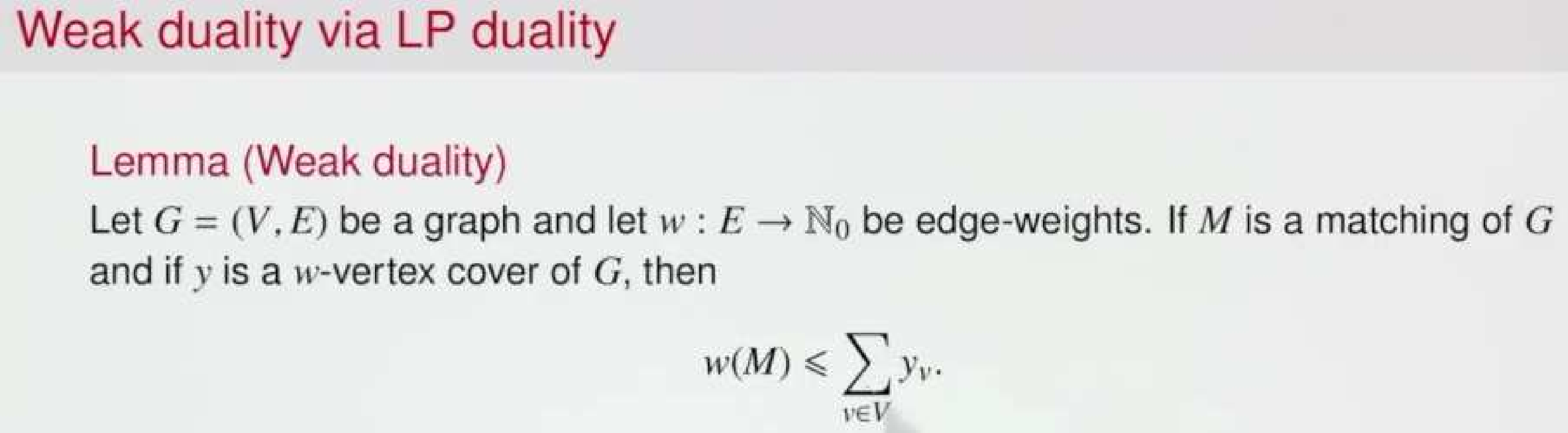
$$\mathbf{y} \in \mathbb{R}^{|\mathbf{V}|}.$$

$$\max w^T x$$

$$\begin{aligned} A^G x &\leqslant \mathbf{1} \\ x &\geqslant 0 \end{aligned}$$

$$\min \mathbf{1}^T y$$

$$\begin{aligned} (A^G)^T y &\geqslant w \\ y &\geqslant 0 \end{aligned}$$

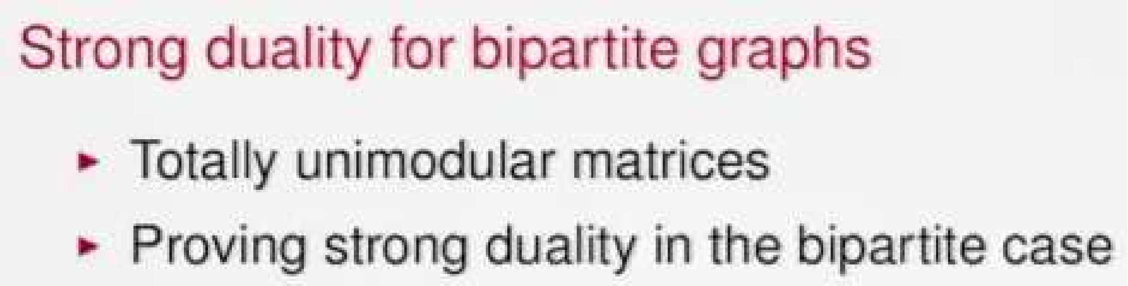


## Weak duality via LP duality

Lemma (Weak duality)

Let $G = (V, E)$ be a graph and let $w : E \to \mathbb{N}_0$ be edge-weights. If $M$ is a matching of $G$ and if $y$ is a $w$-vertex cover of $G$, then

$$w(M) \leqslant \sum_{v \in V} y_v.$$



## Strong duality for bipartite graphs

- Totally unimodular matrices
- Proving strong duality in the bipartite case

# Totally unimodular matrices

A matrix $A \in \{0, \pm 1\}^{m \times n}$ is *totally unimodular*, if the determinant of each square sub-matrix of $A$ is equal to $0, \pm 1$.

Example:

$$\begin{array}{cccc} & \downarrow & & \downarrow \\ \rightarrow & 1 & 1 & 0 \\ \rightarrow & 0 & -1 & 1 \\ & \uparrow & \uparrow\uparrow & \uparrow \end{array}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Which of the two matrices are TU?

$$\begin{array}{ccccc} \rightarrow & 1 & 1 & 1 & 0 \\ \rightarrow & -1 & 0 & 1 & 0 \\ & 1 & 1 & 0 & 1 \\ & \uparrow & & \uparrow & \end{array} \qquad \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

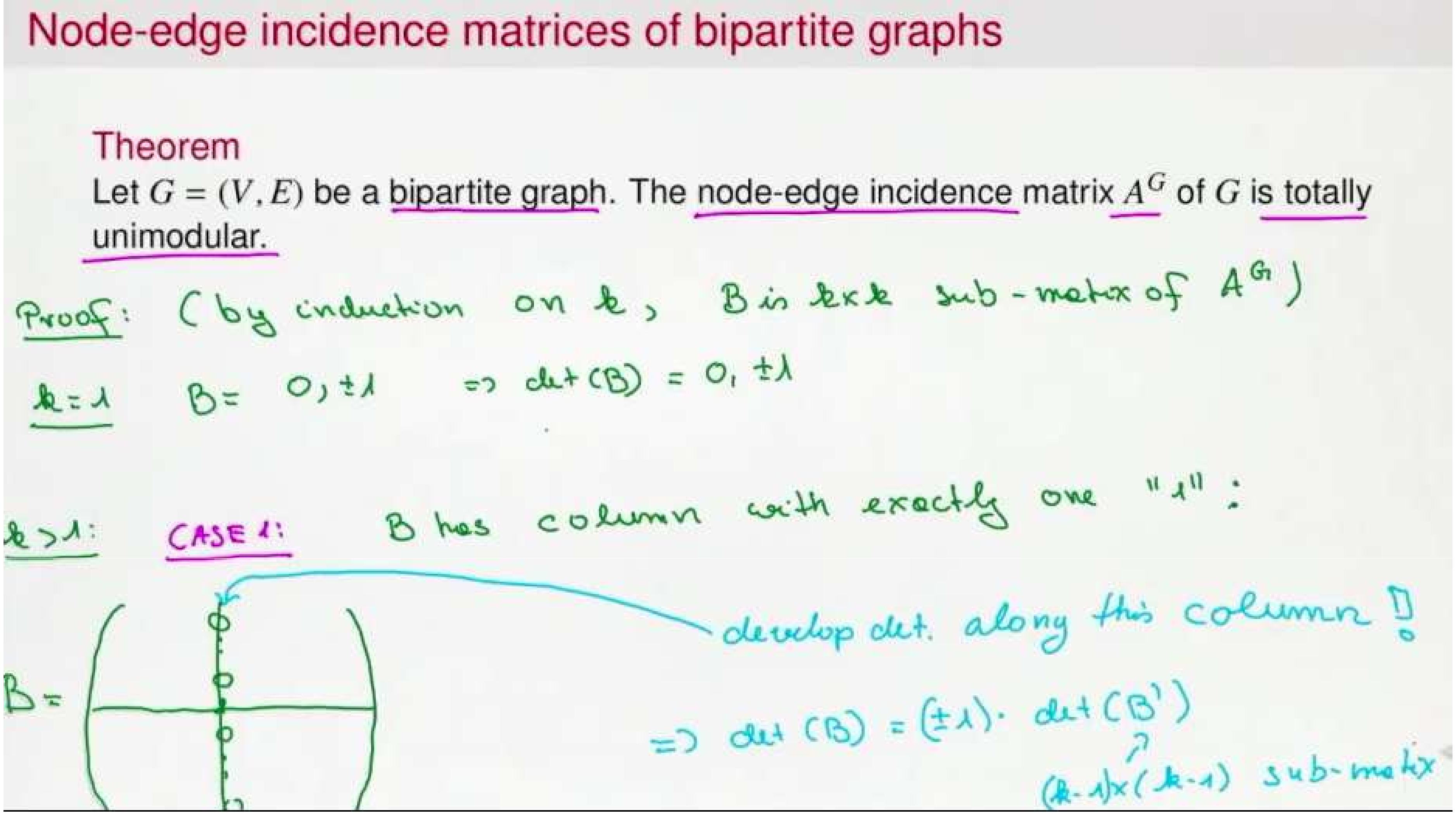
$$\bigcirc \qquad\qquad\qquad \otimes$$

$$\det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2$$

# Node-edge incidence matrices of bipartite graphs

Theorem

Let $G = (V, E)$ be a bipartite graph. The node-edge incidence matrix $A^G$ of $G$ is totally unimodular.



# Node-edge incidence matrices of bipartite graphs

**Theorem**

Let $G = (V, E)$ be a bipartite graph. The node-edge incidence matrix $A^G$ of $G$ is totally unimodular.

Proof: (by induction on $k$, $B$ is $k \times k$ sub-matrix of $A^G$)

$k = 1$  $B = 0, \pm 1$  $\Rightarrow \det(B) = 0, \pm 1$

$k > 1$: CASE 1: $B$ has column with exactly one "1":

$B =$

develop det. along this column !

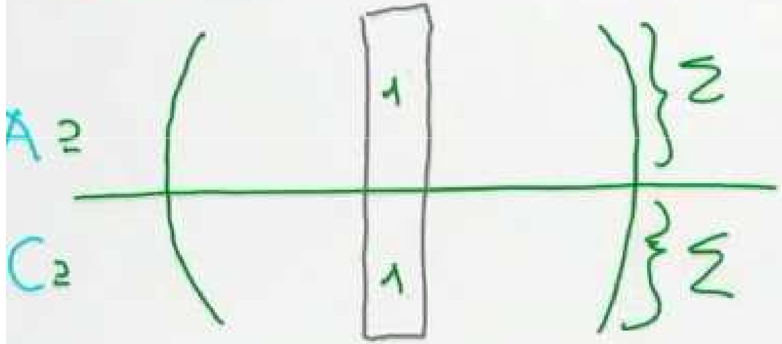
$\Rightarrow \det(B) = (\pm 1) \cdot \det(B')$

$(k-1) \times (k-1)$ sub-matrix

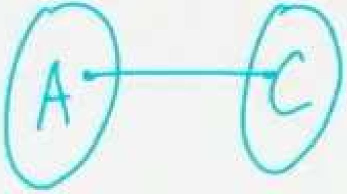
# Node-edge incidence matrices of bipartite graphs

**CASE2:** Each column of $B$ contains exactly 2 "1"s :

ORDER ROWS of $B$ such that Vertices $V = A \dot{\cup} C$ from bi-partition $A$ are on top. (possibly multiplying det by -1)



$$\det(B) = \boxed{0}$$



# Totally unimodular matrices and integer programs

$\max\{c^T.x : Ax \le b, x \ge 0, x \in \mathbb{R}^n\} \equiv \max\{c^T x : Ax \le b, x \ge 0, x \in \mathbb{Z}^n\}$

**Theorem**

If $A \in \mathbb{Z}^{m\times n}$ is totally unimodular and $b \in \mathbb{Z}^m$, then every vertex of the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leqslant b,\ x \geqslant 0\}$ is integral.

Proof:

$$\begin{pmatrix} A \\ -I \end{pmatrix} x \le \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$B \subseteq \{1, \dots, m+n\}$ basis, then

$$B = \underset{\{1,\dots,m\}}{\overset{\cap\!\!\mid}{B_1}} \cup \underset{\{m+1,\dots,m+n\}}{\overset{\cap\!\!\mid}{B_2}}$$

$B_2 = \{i+m : I\}$

$I \subseteq \{1, \dots, n\}$

$x^*_{\bar{I}}$ unique solution of

Quiz: $x^*_I = \boxed{0}$

$\{1,\dots,n\} \setminus I = \bar{I}$

$\tilde{A} \cdot x = \tilde{b}$, where $\tilde{A}$ is $k \times k$ Sub-matrix of A and $\tilde{b}$ is vector having $k$ of the components of b

Quiz: $k = |\bar{I}|$ ☒ $k = n - |I|$ ☒ $k = |I|$ ○

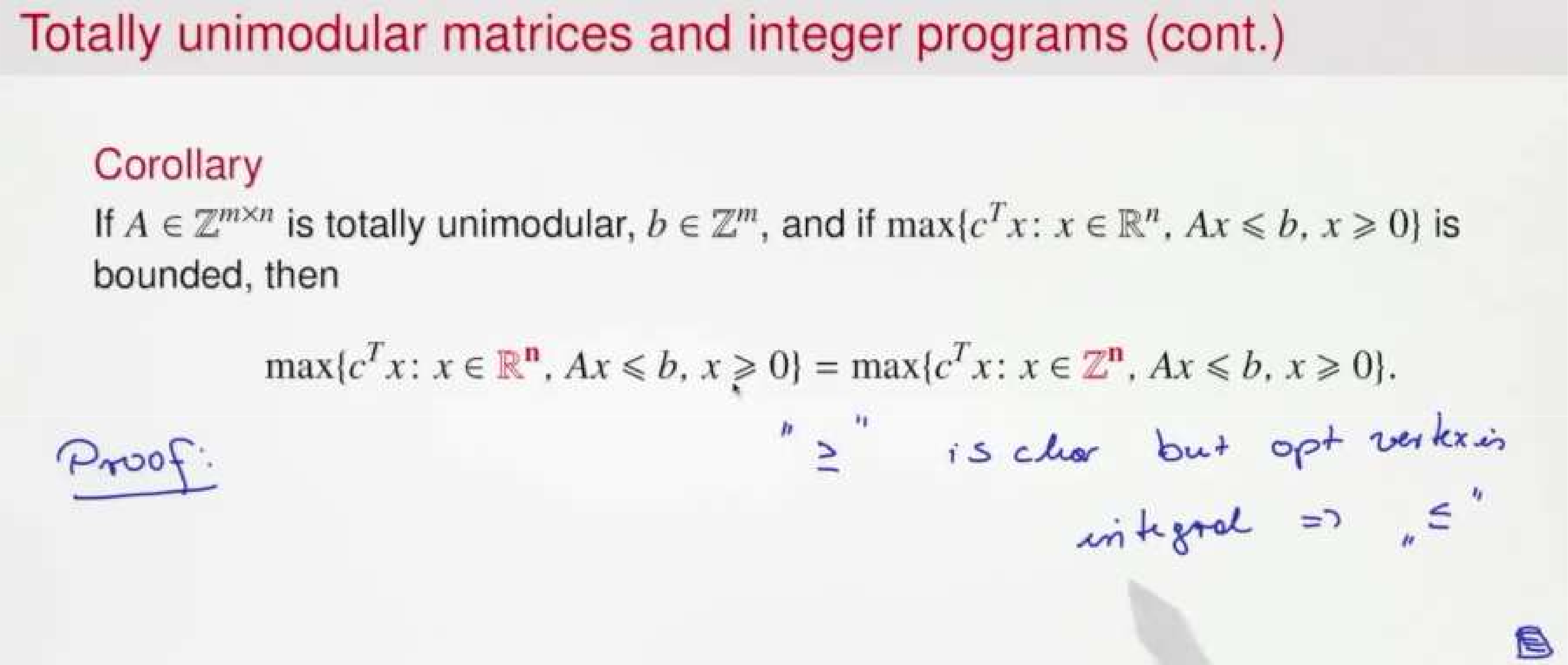
# Totally unimodular matrices and integer programs

Using the matrix inversion formula $\tilde{A}^{-1} = \underset{\in \{\pm 1\}}{\boxed{\frac{1}{\det(\tilde{A})}}} \cdot \operatorname{adj}(\tilde{A})$

$$\operatorname{adj}(\tilde{A}) = \begin{pmatrix} \det(\tilde{A}_{11}) & -\det(\tilde{A}_{21}) & \cdots \\ -\det(\tilde{A}_{12}) & \underset{\in\{0, \pm 1\}}{\boxed{\det(\tilde{A}_{22})}} & \cdots \\ \vdots & & \end{pmatrix}$$

integer matrix

$$x^*_I = \underset{\in \mathbb{Z}^{|I| \times |I|}}{\boxed{\tilde{A}^{-1}}} \cdot \tilde{b} \in \mathbb{Z}^{|I|}$$



# Totally unimodular matrices and integer programs (cont.)

**Corollary**

If $A \in \mathbb{Z}^{m \times n}$ is totally unimodular, $b \in \mathbb{Z}^m$, and if $\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant b,\ x \geqslant 0\}$ is bounded, then

$$\max\{c^T x \colon x \in \mathbb{R}^{\mathbf{n}},\ Ax \leqslant b,\ x \geqslant 0\} = \max\{c^T x \colon x \in \mathbb{Z}^{\mathbf{n}},\ Ax \leqslant b,\ x \geqslant 0\}.$$

Proof:

"$\geq$" is clear but opt vertex is integral $\Rightarrow$ "$\leq$"

# Strong duality in the bipartite case

**Theorem (Egerváry 1931)**

Let $G = (V, E)$ be a bipartite graph and let $w : E \to \mathbb{N}_0$ be edge-weights. The maximum weight of a matching is equal to the minimum value of a $w$-vertex cover.

Proof: Max. weight match — Min value w-vertex cover

$$OPT_{IP} \underset{=}{\le} OPT_{LP} \longleftarrow = \longrightarrow OPT_{LP} \underset{=}{\le} OPT_{IP}$$

$A^G$ is TU

↑ Strong duality

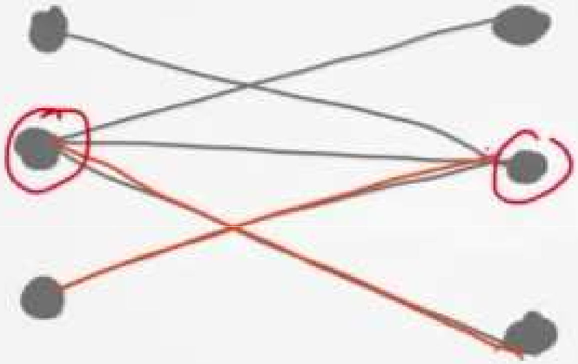
$(A^G)^{\textcircled{T}}$ is TU

$OPT_{IP} = OPT_{IP}$

□

# König's theorem

A *vertex cover* of a graph $G = (V, E)$ is a subset $U \subseteq V$ such that $e \cap U \neq \emptyset$ for each $e \in E$.



W- vertex cover for

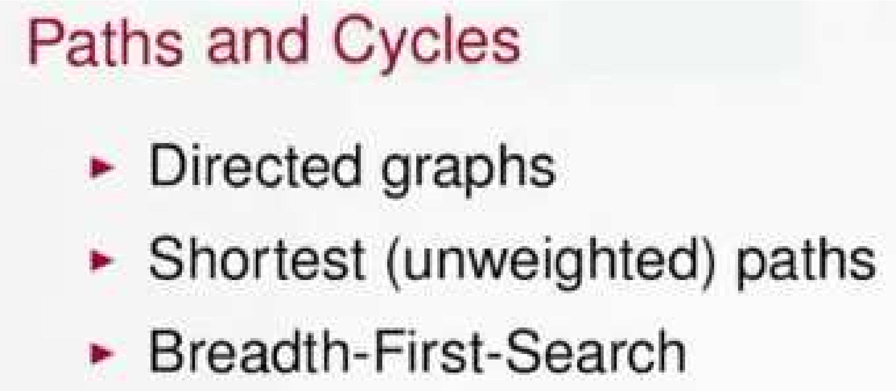
w= 1

<u>Quiz</u>: Prove that max.

cardinality of matching = 2

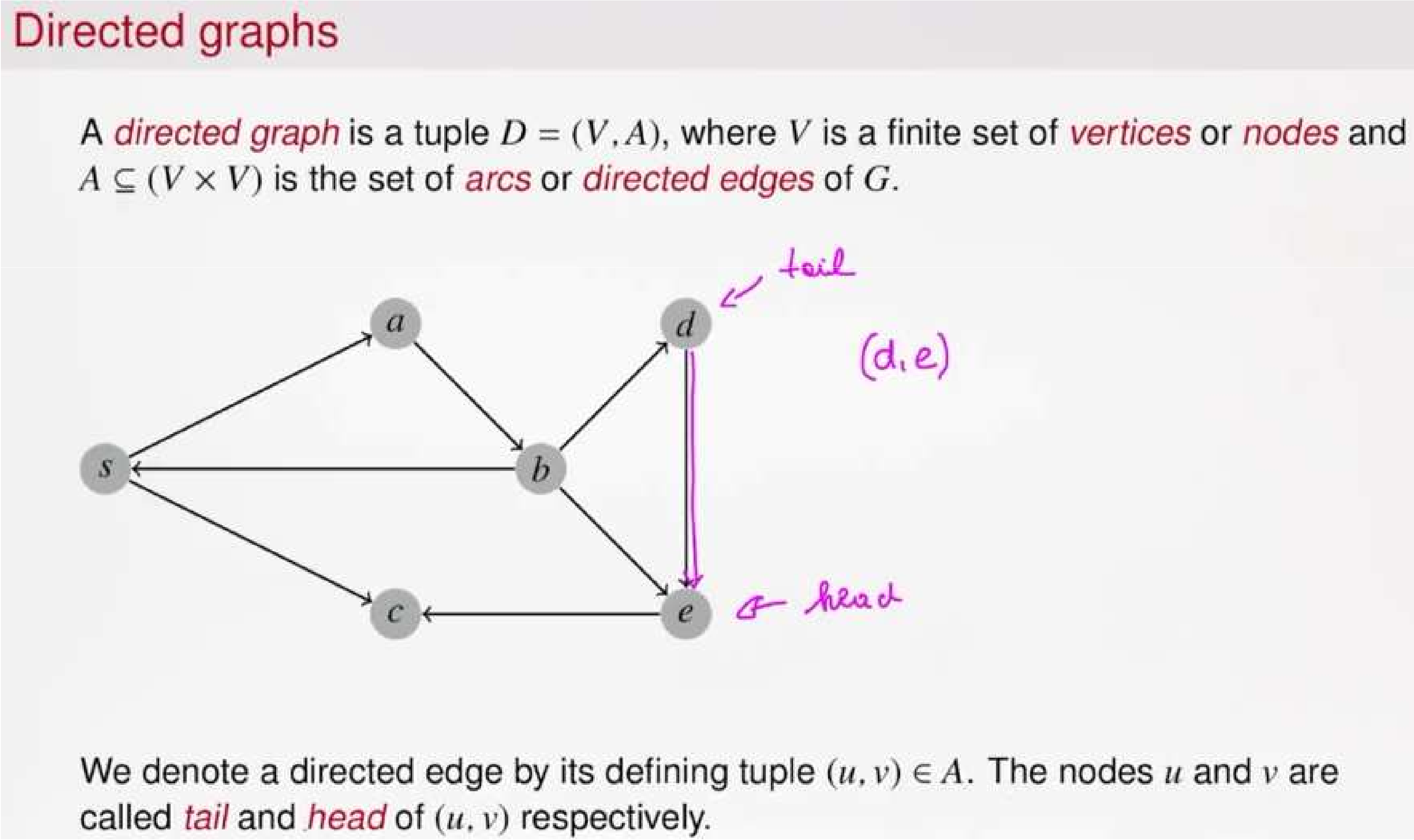
## Theorem (König 1931)

Let $G = (V, E)$ be a bipartite graph. The maximum cardinality of a matching of $G$ is equal to the minimum cardinality of a vertex cover of $G$.



## Paths and Cycles

- Directed graphs
- Shortest (unweighted) paths
- Breadth-First-Search



## Directed graphs

A *directed graph* is a tuple $D = (V, A)$, where $V$ is a finite set of *vertices* or *nodes* and $A \subseteq (V \times V)$ is the set of *arcs* or *directed edges* of $G$.

We denote a directed edge by its defining tuple $(u, v) \in A$. The nodes $u$ and $v$ are called *tail* and *head* of $(u, v)$ respectively.

## Unweighted distance

The *distance* $d(s,t)$ between two nodes $s,t \in V$ is the smallest $k \in \mathbb{N}_0$ such that there exists a path $s = v_0, \dots, v_k = t$. (Possibly $\infty$).
$d(s,t)$ is the length of the *shortest path* connecting $s$ and $t$.

---

## Quiz

What is the largest possible length of a path a directed graph $D = (V,A)$ with $|V| = n$?

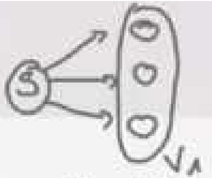
- $n$
- [✓ $n - 1$]
- $n^2 - 1$

Which of the following are upper bounds for the number of directed paths of length $n - 1$ in directed graph with $n$ nodes?

- [✓ $n!$]
- $2^n$
- $n$

PATH SPECIFIED BY Sequence $v_1\ v_2, \dots, v_n$ $n!$

# Distance labels

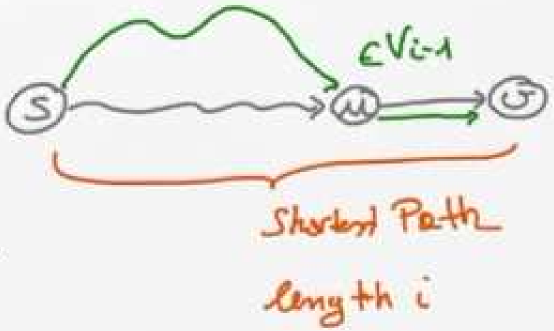


For $i \in \mathbb{N}_0$, $V_i \subseteq V$ denotes the set of vertices that have distance $i$ from $s$. Notice that $V_0 = \{s\}$.

## Proposition

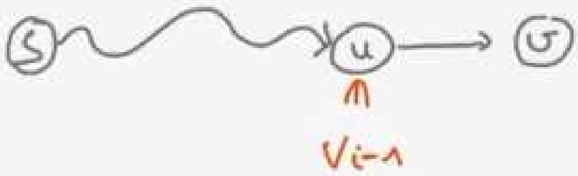
For $i = 1, \ldots, n-1$, the set $V_i$ is equal to the set of vertices $v \in V \setminus (V_0 \cup \cdots \cup V_{i-1})$ such that there exists an arc $(u, v) \in A$ with $u \in V_{i-1}$.

**Proof:** „$\subseteq$" $v \in V_i$



„$\supseteq$" $v \in V_j$ , $j \geq i$

$j \leq i \Rightarrow j = i$



# Analysis

Cheat: Ignore init. of ARRAYS.

With this initialization $O(|V| + |A|)$

## Theorem

The Breadth-First-Search algorithm runs in time $O(|A|)$. It is thus a *linear time* algorithm.

```
while $Q \neq \emptyset$
    $u := head(Q)$
    for each $v \in \delta^+(u)$
        if $(D[v] = \infty)$
            $\pi[v] := u$
            $D[v] := D[u] + 1$
            $enqueue(Q, v)$
    $dequeue(Q)$
```

$|\delta^+(u)|$

Iteration $u$: At most $c_1 \cdot |\delta^+(u)| + c_2$ elementary operations.

$$C_1 \cdot |\delta^+(u)| + C_2$$

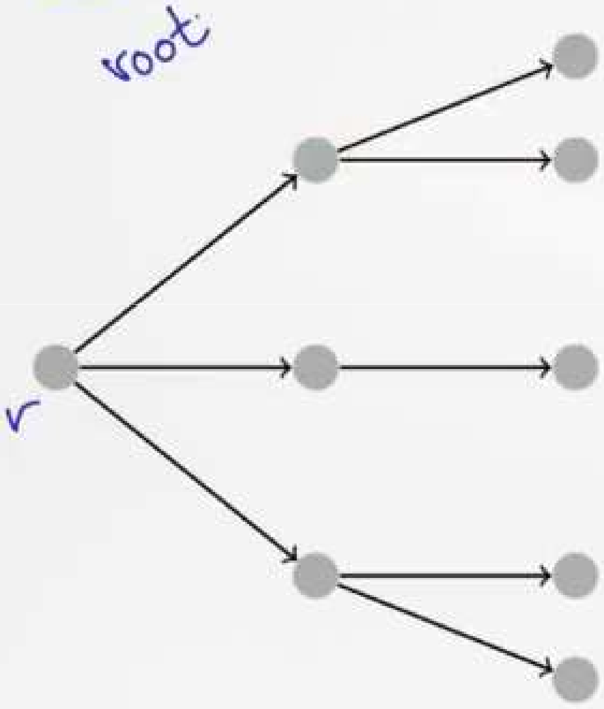
$$\sum_{\substack{u \in V \\ u \text{ reachable} \\ \text{from } s}} C_1 \cdot |\delta^+(u)| + C_2 \leq C_1 \cdot |A| + C_2 |V_{\substack{\text{reach.} \\ \text{from } s}}|$$

$$= O(|A|)$$

elementary op. □

# Directed trees

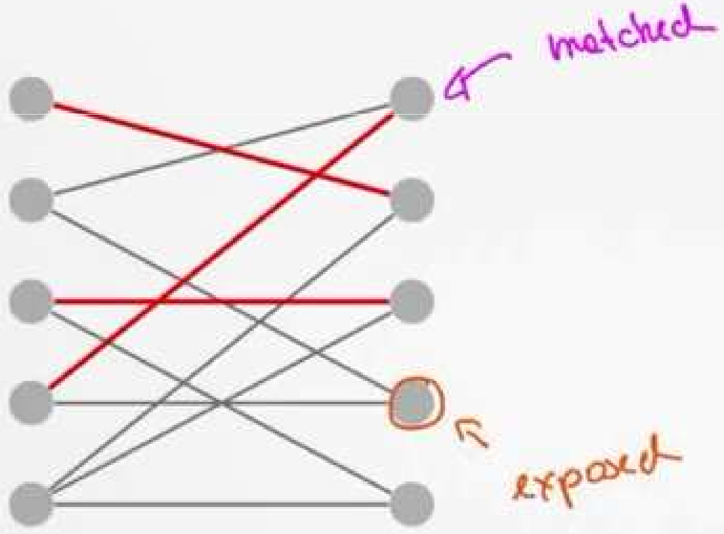
A *directed tree* is a directed graph $T = (V, A)$ with $|A| = |V| - 1$ and there exists a node $r \in T$ such that there exists a path from $r$ to all other nodes of $T$.



## Paths and Cycles

- Maximum cardinality bipartite matchings
- Augmenting paths
- An $O(m \cdot n)$ algorithm

## Exposed and matched nodes



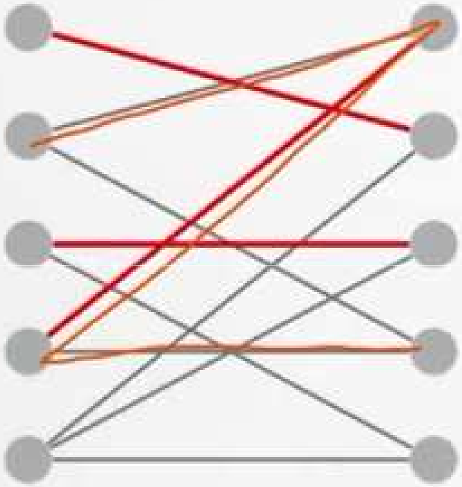
Let $G = (V, E)$ be an undirected bipartite graph.
We are interested in a matching of max. cardinality.

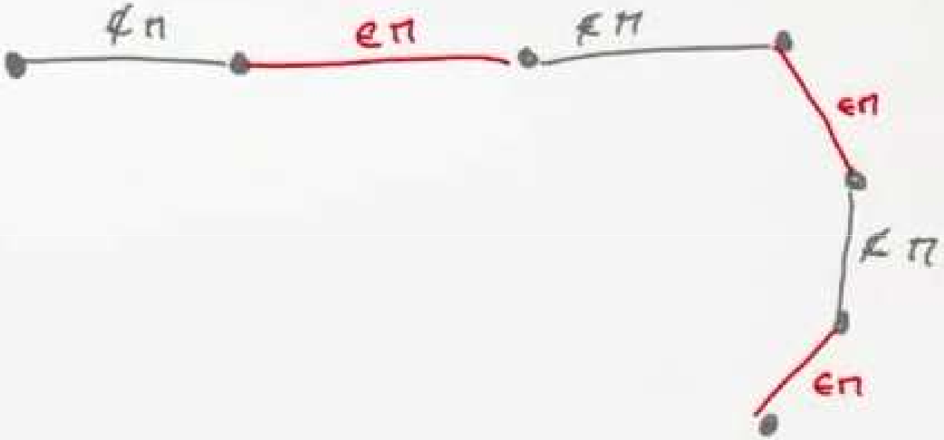
Let $M \subseteq E$ be a matching.

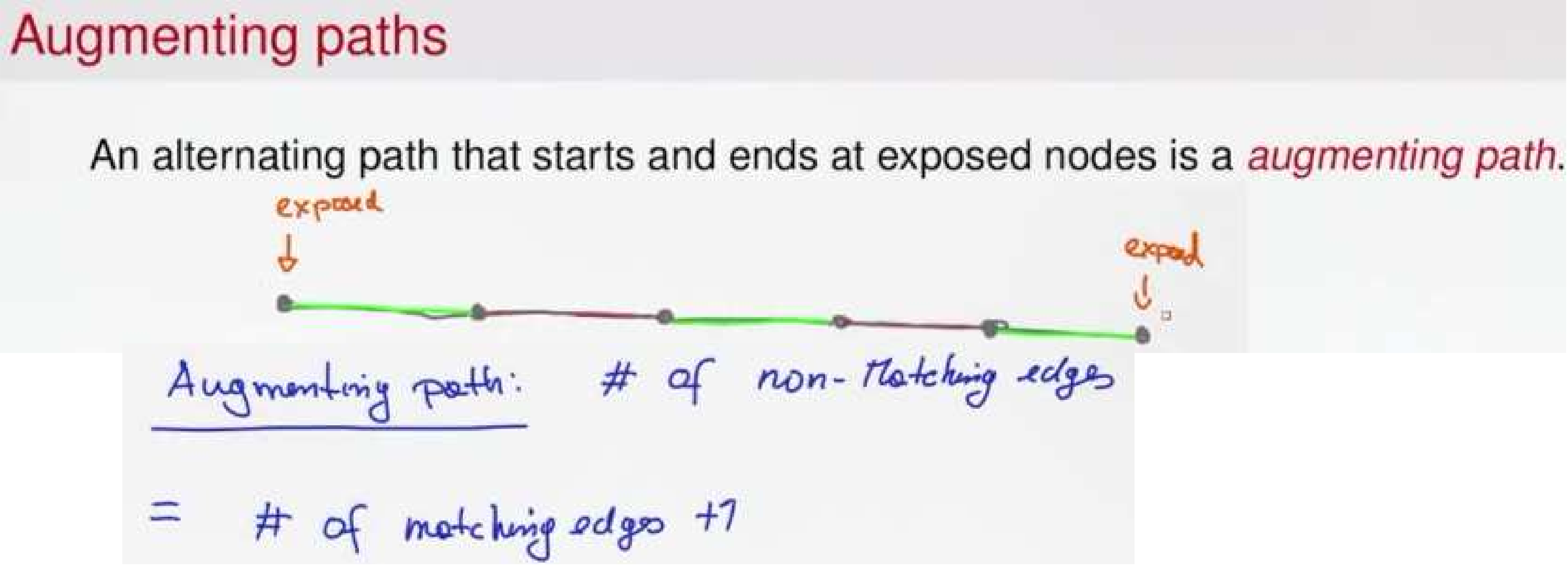
- A vertex that is an endpoint of an edge in $M$ is *matched*.
- A non-matched vertex is *exposed*

# Alternating paths

An alternating path with respect to a matching $M$ is a path that alternates between edges in $M$ and edges in $E \setminus M$.





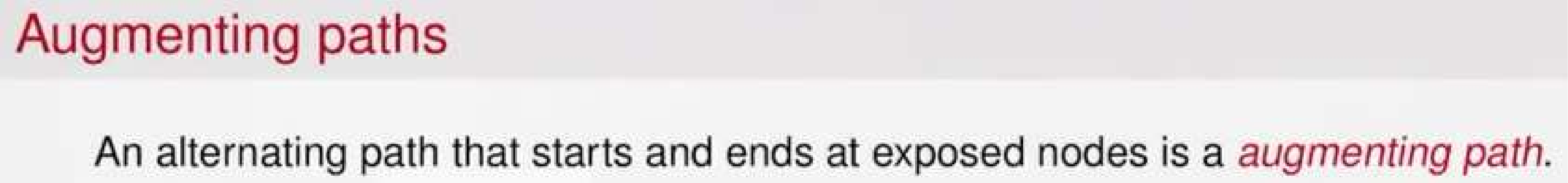


## Augmenting paths

An alternating path that starts and ends at exposed nodes is a *augmenting path*.

Augmenting path: # of non-Matching edges

= # of matching edges +1



## Augmenting paths

An alternating path that starts and ends at exposed nodes is a *augmenting path*.

# A criterion for maximal cardinality

## Theorem

A matching $M$ of a (not necessarily bipartite) graph is of maximum cardinality if and only if there are no augmenting paths with respect to $M$.

Proof: "$\Rightarrow$"

$$M' = M \setminus (E(P) \cap M) \cup (E(P) \setminus M) = M \triangle E(P)$$

$$= (M \cup E(P)) \setminus (M \cap E(P))$$

$|M'| = |M| + 1$ ↯

# A criterion for maximal cardinality

**Theorem**

A matching $M$ of a (not necessarily bipartite) graph is of maximum cardinality if and only if there are no augmenting paths with respect to $M$.

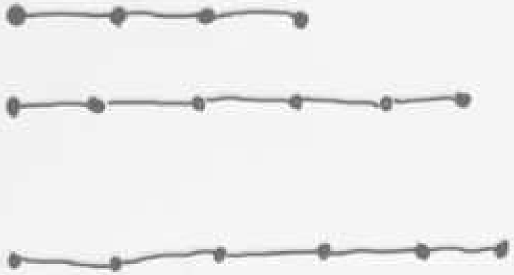
"$\Leftarrow$" Assume there exists Matching $M'$ with $|M'| > |M|$. Consider $G' = (V, M' \Delta M)$.

$\deg(v) = |\delta(v)|$

$= 3$ Possible ?

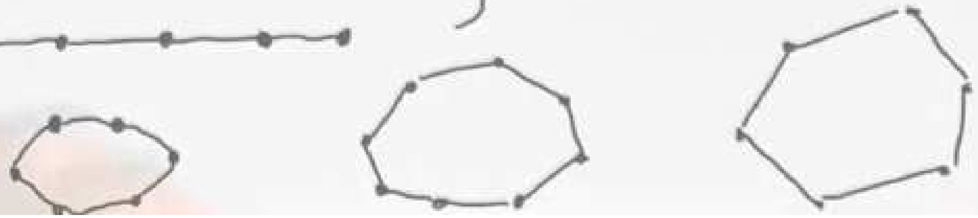
YES ○

NO ~~○~~



Node disjoint paths

$\forall v \in V: \deg(v) \leq 2.$ (in $G'$)



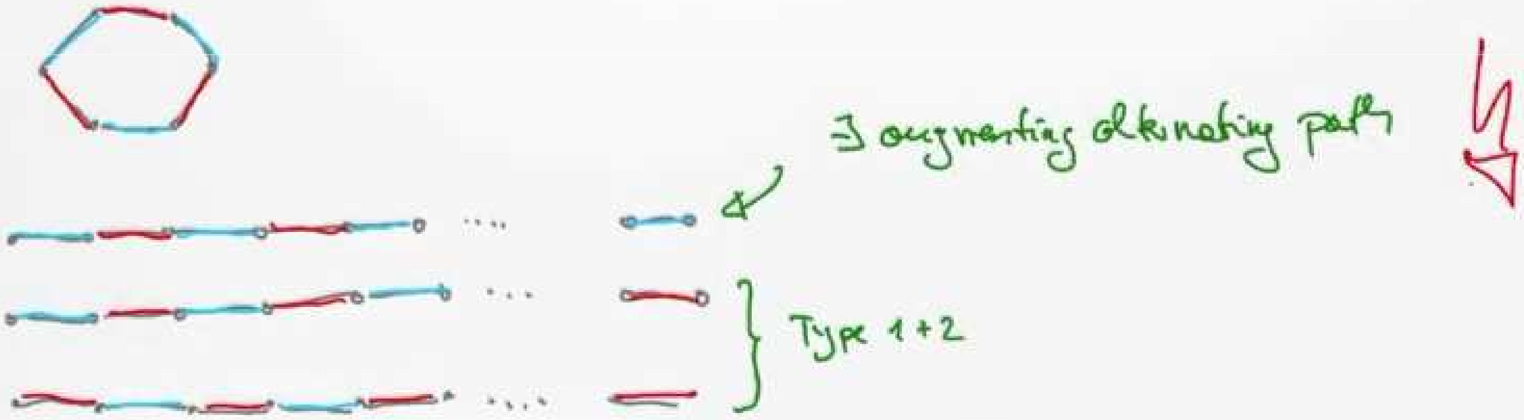
Node disjoint cycles.

# A criterion for maximal cardinality

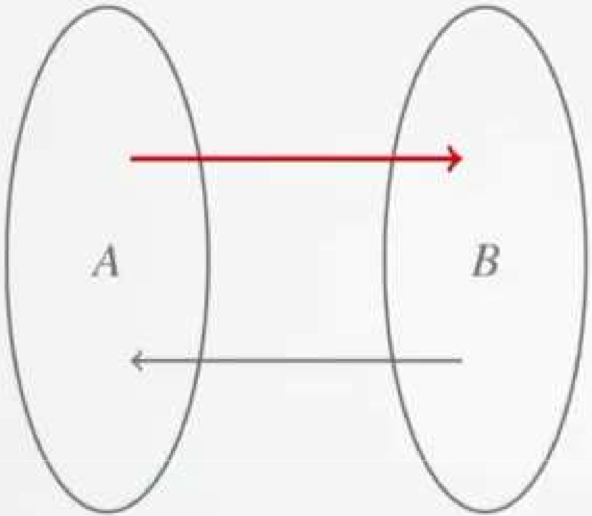
**Theorem**

A matching $M$ of a (not necessarily bipartite) graph is of maximum cardinality if and only if there are no augmenting paths with respect to $M$.

"$\Leftarrow$" Assume there exists Matching $M'$ with $|M'| > |M|$. Consider $G' = (V, M' \triangle M)$.



# Computing augmenting paths

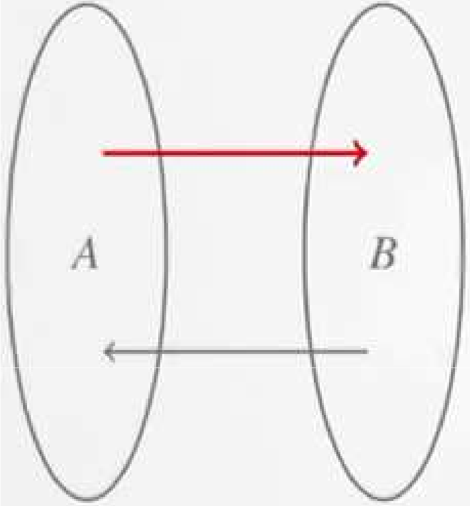


- Turn $G = (A + B, E)$ into a directed graph $D = (V, A)$ as follows.
- Direct an edge in the matching from $A$ to $B$.
- Direct an edge in $E \setminus M$ from $B$ to $A$.
- Find a path in this directed graph between two exposed nodes.

Quiz: Such a path starts with an exposed node in B and ends in an exposed node in A

Type A or B at appropriate place.

# Computing augmenting paths (cont.)

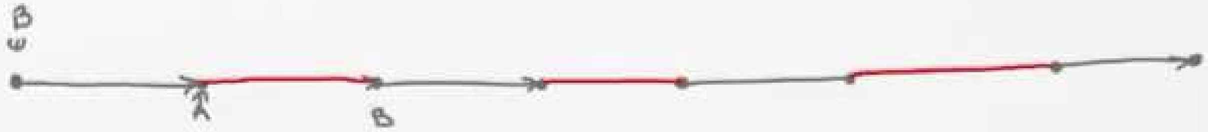


**Theorem**

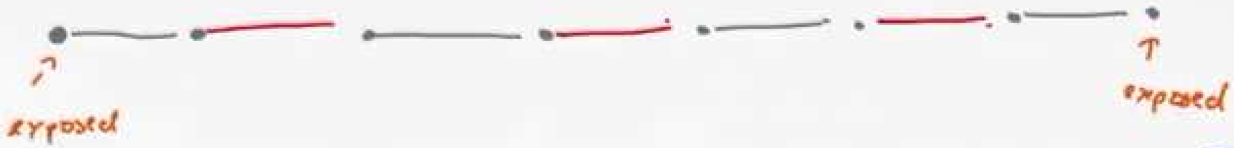
There exists an augmenting path in $G$ for $M$ if and only if there exists a path from an exposed node in $B$ to an exposed node in $A$ in the directed graph $D$.

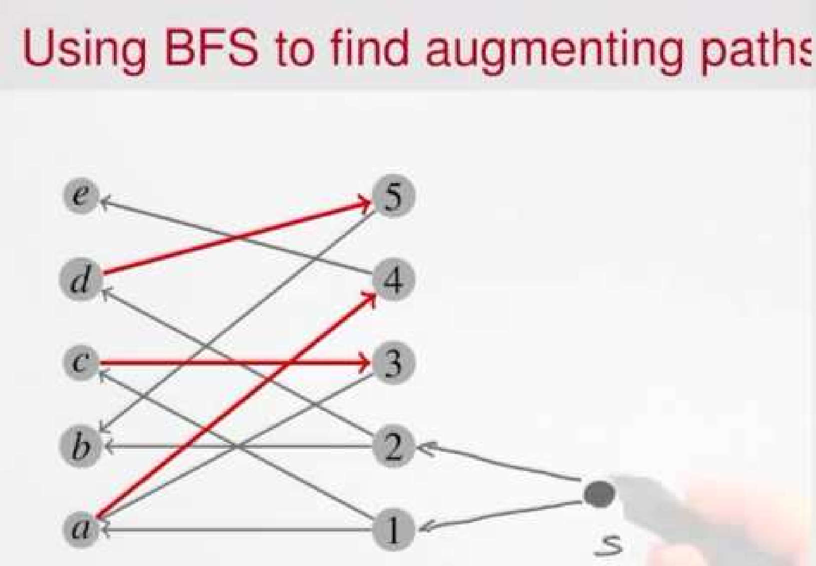
Proof:

„⇒"

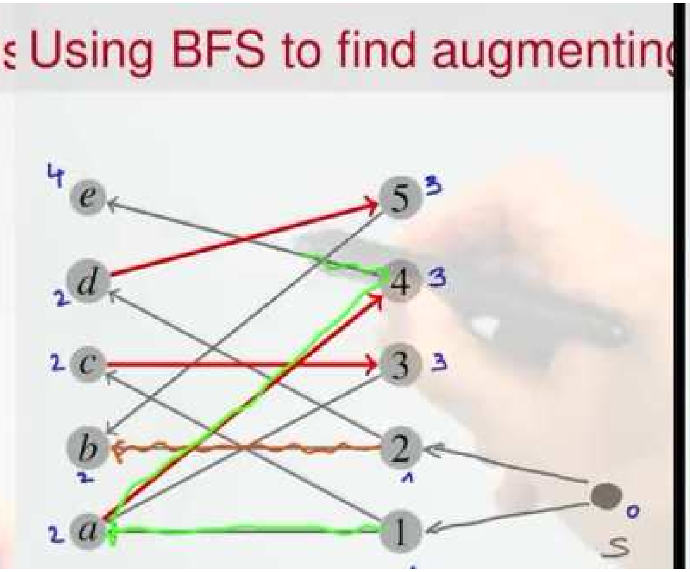


„⇐"

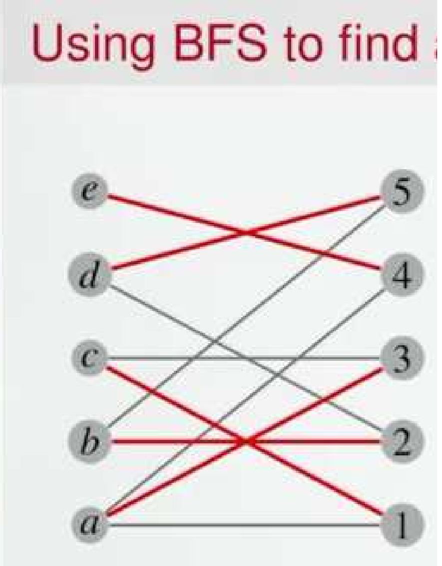




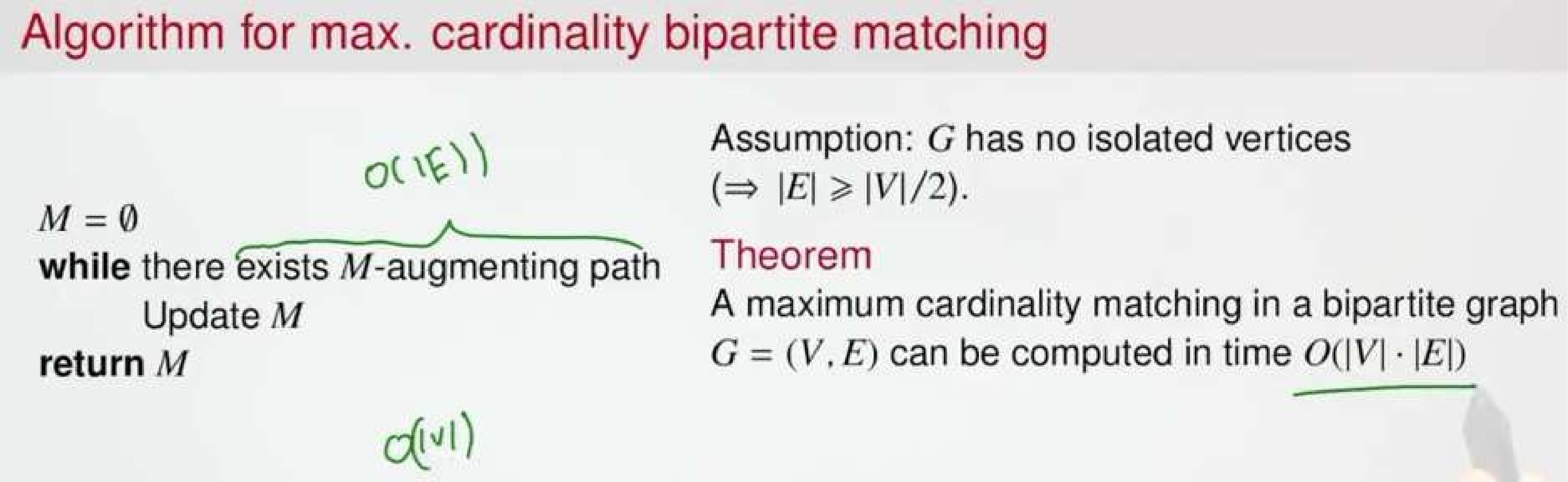
Using BFS to find augmenting paths



Using BFS to find augmenting



Using BFS to find



## Algorithm for max. cardinality bipartite matching

O(|E|)

```
$M = \emptyset$
while there exists $M$-augmenting path
        Update $M$
return $M$
```

O(|V|)

Assumption: $G$ has no isolated vertices
($\Rightarrow$ $|E| \geqslant |V|/2$).

**Theorem**

A maximum cardinality matching in a bipartite graph $G = (V, E)$ can be computed in time $\underline{O(|V| \cdot |E|)}$

## Paths and Cycles

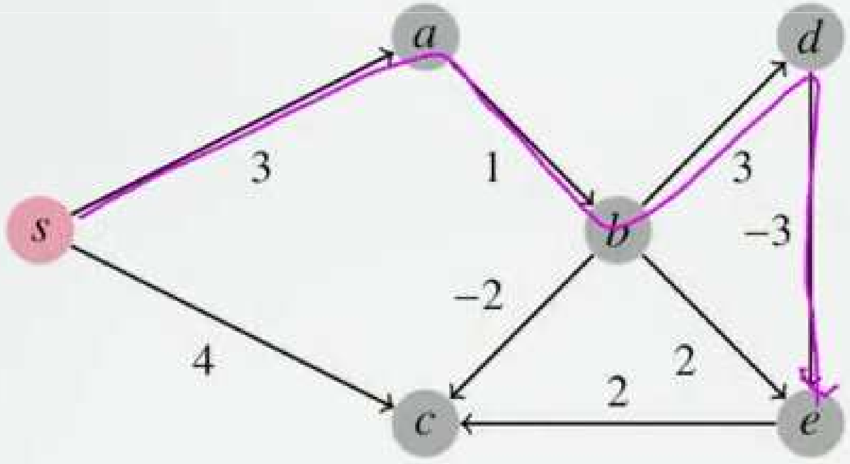
- Weighted directed graphs
- Shortest paths
- Bellman-Ford Algorithm

# Weighted directed graphs

Let $D = (V, A)$ be a directed graph (without self loops). Let $\ell : A \to \mathbb{R}$ be the *lengths* of the arcs. The *length* of a walk $W = v_0, \dots, v_k$ is the sum of the lengths of its arcs:

$$\ell(W) = \sum_{i=1}^{k} \ell(v_{i-1}, v_i).$$

The *distance* between two nodes $s$ and $t$ is the length of a *shortest path* from $s$ to $t$.



s, a, b, c length = 2

$d_\ell(s,t)$

$d_\ell(s,e) = \boxed{4}$

# Shortest path problem

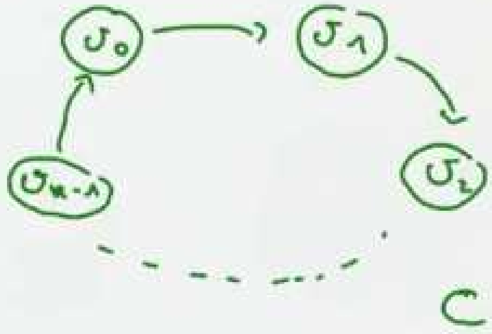
## The shortest path problem (single source)

Given a directed graph with edge lengths and a designated node $s$, compute $d(s, v)$ for each $v \in V$.

- Is NP-complete in general.
- Can be solved in polynomial time, if there are negative cycles.

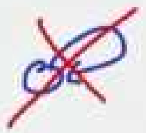
A *cycle* is a walk $v_0, v_1, \ldots, v_k$ with $v_0 = v_k$.

$$\ell(C) = \sum_{i=0}^{k-1} \ell(v_i, v_{\underbrace{i+1}_{\text{mod } k}})$$



# The Bellman-Ford method

A method to compute minimum length *walks*.



*Given:* $D = (V, A)$ (no self-loops), $\ell : A \to \mathbb{R}$ and designated node $s \in V$

*Goal:* Compute shortest path distances form $s$ to all other nodes

*Assumption:* Each node is reachable from $s$

# The Bellman-Ford method (cont.)

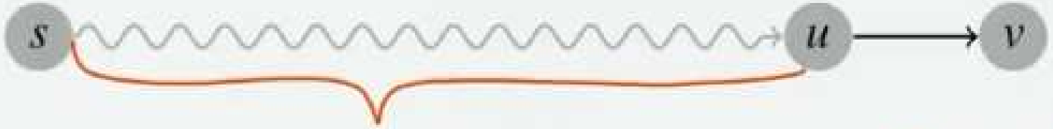
For $k \geqslant 0$ and $t \in V$:

$d_k(t) =$ minimum length of any $s - t$ walk, traversing at most $k$ arcs. (possibly $\infty$)

Suppose $d_i(t)$ is known for each $i \leqslant k$ and each $t \in V$.

*Now*: Compute $d_{k+1}(t)$: for each $t \in V$.

Case 1: ~~The~~ A shortest walk traversing at most $k+1$ arcs traverses exactly $k+1$ arcs.



$d_k(u)$

$d_k(u) + \ell(u,v)$

Case 2: ~~The~~ A shortest walk traversing at most $k+1$ arcs traverses at most $k$ arcs.

$d_{k+1}(t) = d_k(t)$

## The Bellman-Ford method (cont.)

$$d_o(s) = 0, \quad d_0(t) = \infty, t \neq s$$

$$k \geqslant 0,\ t \in V : d_{k+1}(t) = \min\{d_k(t), \min_{(u,t)\in A}\{d_k(u) + \ell(u,t)\}.$$

Procedure to compute the values $d_{k+1}(t)$ *assuming* values $d_k(t)$ are pre-computed:

```
for each $t \in V$:
        $d_{k+1}(t) := \underline{d_k(t)}$

for each $(u, t) \in A$
        if: $d_k(u) + \ell(u, t) < d_{k+1}(t)$
            $d_{k+1}(t) := \underline{d_k(u) + \ell(u, t)}$
```

Valid upperbounds for $d_{k+1}(t)$

Correct !

# Negative cycles

## Theorem

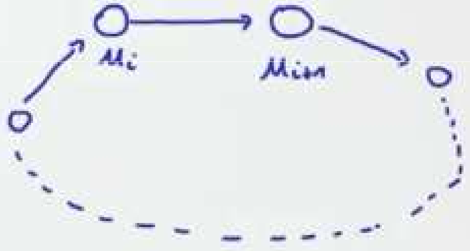
Given $D = (V, A)$, $s \in V$, $\ell : A \to \mathbb{R}$, one has $d_n = d_{n-1}$ for $n = |V|$ iff $D$ does not have a cycle of negative length that is reachable from $s$.

<u>Proof:</u> „$\Rightarrow$" Suppose $u_0, u_1, u_2, \dots, u_k, u_0$ is a cycle reachable from $s$

$d_{n-1}(u_i) < \infty \quad \forall i = 0, \dots, k$

$$\boxed{0} = \sum_{i=0}^{k} \overbrace{d_n(u_{\underset{\text{mod}(k)}{i+1}})}^{\le d_{n-1}(u_i) + \ell(u_i, u_{i+1})} - \underbrace{d_n(u_i)}_{= d_{n-1}(u_i)}$$

$$\le \sum_{i=0}^{k} \ell(u_i, u_{i+1}) = \ell(C)$$

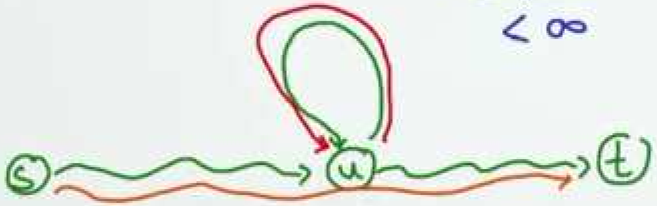


# Negative cycles

## Theorem

Given $D = (V, A)$, $s \in V$, $\ell : A \to \mathbb{R}$, one has $d_n = d_{n-1}$ for $n = |V|$ iff $D$ does not have a cycle of negative length that is reachable from $s$.

"$\Leftarrow$" Suppose $\underbrace{d_n(t)}_{<\infty} < \underbrace{d_{n-1}(t)}_{<\infty}$ $\Rightarrow$ $t$ is reachable from $s$



length of shortest s-t walk using exactly n arcs is < length of any s-t walk using n-1 arcs

$W_1$ $s = w_0, w_1, \ldots, \overset{\overset{u}{\shortparallel}}{w_i}, \ldots, \overset{\overset{u}{\shortparallel}}{w_j}, \ldots, w_n = t$

$W_2$ $s = w_0, w_1, \ldots, w_i, w_{j+1}, \ldots w_n = t$

$C$ $w_i, w_{i+1}, \ldots, w_j$

# Negative cycles

**Theorem**

Given $D = (V, A)$, $s \in V$, $\ell : A \to \mathbb{R}$, one has $d_n = d_{n-1}$ for $n \geqq |V|$ iff $D$ does not have a cycle of negative length that is reachable from $s$.

$W_1$ $\quad s = w_0, w_1, \dots, \overset{\overset{u}{\shortparallel}}{w_i}, \dots, \overset{\overset{u}{\shortparallel}}{w_j}, \dots, w_n = t$

$W_2$ $\quad s = w_0, w_1, \dots, w_i, w_{j+1}, \dots w_n = t$

$C$ $\quad w_i, w_{i+1}, \dots, w_j$

$$\ell(W_1) = \underbrace{\sum_{\mu=0}^{i-1} \ell(w_\mu, w_{\mu+1})} + \overbrace{\sum_{\mu=i}^{j-1} \ell(w_\mu, w_{\mu+1})}^{\ell(C) < 0} + \underbrace{\sum_{\mu=j}^{n-1} \ell(w_\mu, w_{\mu+1})}$$

$$+$$

$$\ell(W_2)$$

## Shortest paths

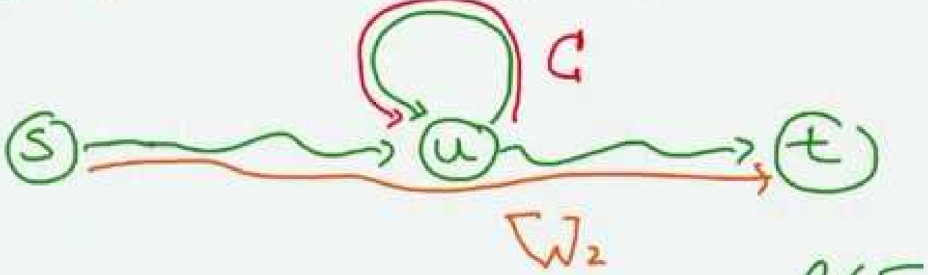
**Theorem**

Given $D = (V, A)$, $s \in V$, $\ell : A \to \mathbb{R}$, and suppose that no negative cycle is reachable from $s$. Then for each $t \in V$ $\underline{d_{n-1}(t)}$ is the distance between $s$ and $t$.

proof: Suppose $d_{n-1}(t) <$ length of shortest path from s to t.

Let $W$ be a shortest walk from s to t using at most n-1 arcs and with a minimal number of arcs.

$W$ :



$\ell(W) = \ell(W_2) + \ell(C)$

$\ell(W) < \ell(W_2)$ .

$\Rightarrow \ell(C) < 0$ ↯

# Computing shortest paths

Compute the values $d_{k+1}(t)$ and the predecessor $\pi_{\underline{k+1}}(t)$ *assuming* values $d_k(t)$ and $\pi_k(t)$ have been pre-computed:

```
for each $t \in V$:
        $d_{k+1}(t) := d_k(t)$
        $\pi_{k+1}(t) := \pi_k(t)$

for each $(u, t) \in A$
        if: $d_k(u) + \ell(u, t) < d_{k+1}(t)$
            $d_{k+1}(t) := d_k(u) + \ell(u, t)$
            $\pi_{k+1}(t) := \boxed{u}$
```

# The shortest path tree

No neg. cycles!

## Theorem

Let $D = (V, A)$ be a directed graph and suppose that each node is reachable from $s$. The directed graph $T = (V, A')$ with $A' = \{(\underset{n-1}{\pi}(u), u) : u \in V \setminus \{s\}\}$ is a directed tree with root $s$. The unique path from $s$ to any vertex $t$ in $T$ is a shortest path from $s$ to $t$ in $D$.

# Running time of Bellman-Ford

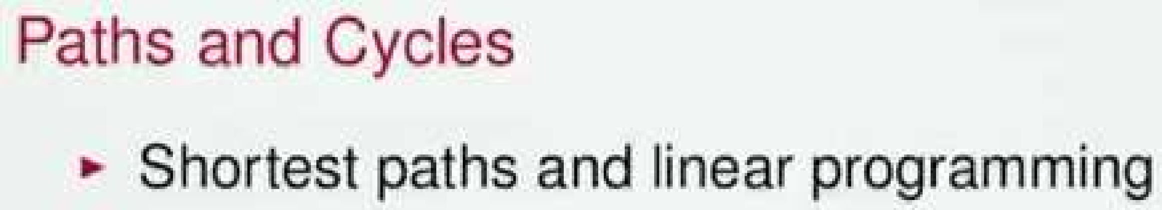
$O(|V|\cdot|A|)$

$O(|V|(|V|+|A|))$

$|A| = \Omega(|V|)$

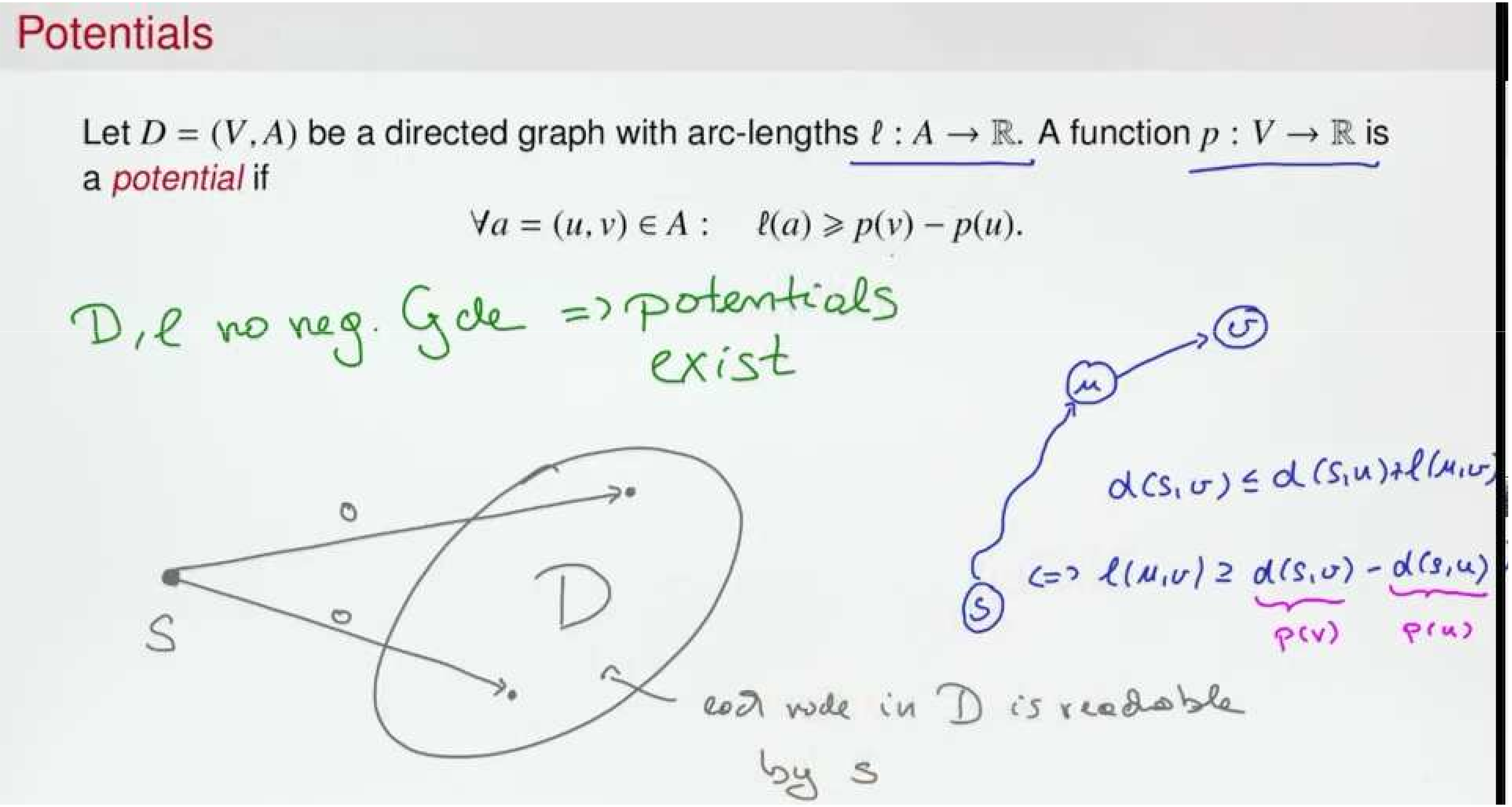
```
initialize
    $\forall t \in V \setminus \{s\}, d_o(t) = \infty,\ \pi_0(t) = 0$        } $O(|V|)$
    $d_0(s) = 0$
for $k = 1$ to $n$
    for each $t \in V$:                                                      } $O(|V|)$
            $d_{k+1}(t) := d_k(t)$
            $\pi_{k+1}(t) := \pi_k(t)$

    for each $(u, t) \in A$                                                  } $O(|A|)$
            if: $d_k(u) + \ell(u, t) < d_{k+1}(t)$
                $d_{k+1}(t) := d_k(u) + \ell(u, t)$
                $\pi_{k+1}(t) := u$
                                                                             } $O(|V|)$
if  $\exists t \in V$ with $d_n(t) < d_{n-1}(t)$                             } $O(|V|)$
    $D$ has negative cycle
```



## Paths and Cycles

- Shortest paths and linear programming



## Potentials

Let $D = (V, A)$ be a directed graph with arc-lengths $\ell : A \to \mathbb{R}$. A function $p : V \to \mathbb{R}$ is a *potential* if

$$\forall a = (u, v) \in A : \quad \ell(a) \geqslant p(v) - p(u).$$

D, ℓ no neg. Cycle => potentials exist

$d(s,v) \leq d(s,u) + \ell(u,v)$

$\Leftrightarrow \ell(u,v) \geq \underbrace{d(s,v)}_{p(v)} - \underbrace{d(s,u)}_{p(u)}$

each node in D is reachable by s

# Existence of potentials

**Theorem**

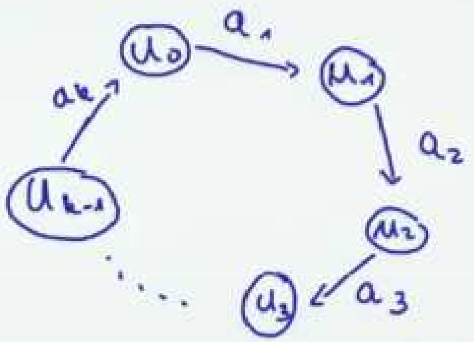
$D = (V, A)$ with $\ell : A \to \mathbb{R}$ has a potential $p$ if and only if each directed cycle is of non-negative length.

$\Leftarrow$ ''

'' ''

$\Rightarrow$

''



$$\ell(C) = \sum_{i=1}^{k} \underbrace{\ell(a_i)}_{\geq p(\underset{\uparrow \text{mod } k}{u_i}) - p(u_{i-1})}$$

$$\geq 0$$

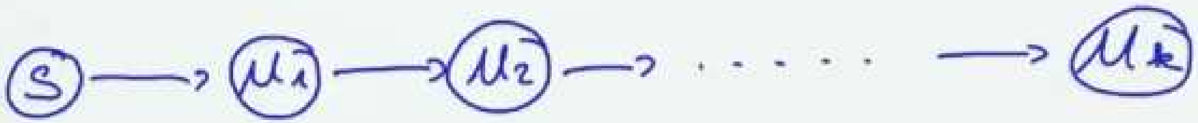
# Computing distances with linear programming

**Theorem**

Let $D = (V, A)$ be a directed graph with arc-lengths $\ell : A \to \mathbb{R}$, $s \in V$ such that each vertex in $V$ is reachable from $s$ and suppose that each directed cycle is non-negative. Let $p$ be a potential with $p(s) = 0$ and $\sum_{v \in V} p(v)$ maximal. Then

$$\forall t \in V : \, p(t) = \mathrm{dist}_\ell(s, t).$$

proof: Shortest path distances are a potential



$p(u_1) \leq \ell(s, u_1)$

$p(u_2) \leq \ell(u_1, u_2) + p(u_1) \leq \ell(u_1, u_2) + \ell(s, u_1)$

$\vdots$

$p(u_k) \leq$ length of PATH.

$p(u) \leq d(s, u)$